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Conditional Probabilities with Evolving Observables and the Problem of Time In Quantum Gravity

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INTRODUCTION

There is by now extensive literature addressing the problem of time in classical and quantum gravity (e.g. Kuchař ‘s review).

The heart of the problem lies in the fact that Einstein’s theory is a totally constrained system whose *Hamiltonian vanishes*, and since observable quantities are those that commute with the constraints (Dirac Observables) they therefore *do not evolve*.

We will discuss here two approaches to this problem.

Both have in common their relational character. In fact, one of the basic ingredients in the different proposals to describe evolution is the use of *relations* between different degrees of freedom in the theory .

- *Evolving Dirac observables.* (Bergmann, DeWitt, Rovelli, Marolf…)
- *Conditional probabilities approach* proposed by Page and Wootters.

We will see that both approaches present problems and do not provide a completely satisfactory solution to the issue of the evolution.

Problems are particularly acute when we try to compute propagators or assign probabilities to histories.

We will show that **a combination of both approaches** addresses most of the issues mentioned above.

1) Evolving Dirac Observables in totally constrained systems:

$$S = \int [p_a \dot{q}^a - \mu^\alpha \phi_\alpha(q,p)] d\tau$$

In the case of GR the constraints are first class

$$\phi_\alpha(q,p) = 0$$

$$\{\phi_\alpha(q,p), \phi_\beta(q,p)\} = C^\gamma_{\alpha\beta} \phi_\gamma(q,p)$$

$$H_T = \mu^\alpha \phi_\alpha(q,p)$$

The Hamiltonian vanishes: the generator of the evolution also generates gauge transformations

Dirac observables are gauge invariant quantities

$$\{O(q,p), \phi_\beta(q,p)\} \approx 0 \qquad \{O(q,p), H_T(q,p)\} \approx 0$$

Therefore, they are constants of the motion.

The issue of time: If the physically relevant quantities in totally constrained systems as general relativity are constants of the motion, how can we describe the evolution?

a) Gauge fixing: $\tau = f(q, p), \quad \tau = q^0$

Great example: ADM.

b) Evolving Dirac observables: Bergmann, DeWitt, Rovelli, Marolf …

$$\{Q_i(t), \phi_\alpha\} \approx 0 \qquad\qquad Q_i(t, q^a, p_a)\big|_{t=q^1} = q_i$$

For instance, for the relativistic particle. $\phi = p_0^2 - p^2 - m^2$

Two independent observables:

$$p, X \equiv q - \frac{p}{\sqrt{p^2 + m^2}} q^0, \qquad Q(t, q^a, p_a) = X + \frac{p}{\sqrt{p^2 + m^2}} t$$

$$Q(t = q^0, q^a, p_a) = q$$

Notice that one needs to assume that there are variables as q^0 that are physically observable, even though they are not Dirac observables

The Quantum Evolving Observables.

Let us consider the elementary case of a non-relativistic free particle

$$S=\int d\tau\left[p_0\dot{x}^0+p\dot{x}-N\left(p_0+\frac{p^2}{2m}\right)\right]$$

$$\mathcal{H}_{aux} = L^2(\mathrm{R}^2, dp_0 dp)$$

with self-adjoint Dirac observables $\{\hat{q}:=\hat{x}-x^0 p/m, \hat{p}, \hat{X}(T):=\hat{q}+\hat{p}\hat{T}/m\}.$

$$X(T)\,|_{T=x^0}=x$$

$$\psi_{p_1}(p,p_0)=\delta\left(p_0+\frac{p^2}{2m}\right)\delta(p-p_1),$$

And eigenvectors
(which solve the constraint as well)

$$\psi_{q_1}(p,p_0)=\delta\left(p_0+\frac{p^2}{2m}\right)\exp ipq_1,$$

$$\psi_{x_1,T}(p,p_0)=\delta\left(p_0+\frac{p^2}{2m}\right)\exp i\left(px_1-\frac{p^2}{2m}T\right).$$

It is now possible to introduce an inner product in the space of solutions of the constraint $f(p,p_0)\delta(p_0+\frac{p^2}{2m})$ and define H_{phys}

$$<\psi_1|\psi_2>_{phys}=\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}dp_0\delta(p_0+\tfrac{p^2}{2m})f^*{}_1(p,p_0)f_2(p,p_0)=\int_{-\infty}^{\infty}dp\tilde{f}_1^*(p)\tilde{f}_2(p)$$

Summarizing, the choice of clock variable $T = x^0$ leads to the standard form of the quantum free particle in the Heisenberg representation. In particular, the transition amplitude is:

$$< \psi_{x,T}|\psi_{x',T'} >= [\frac{2\pi i(T-T')}{m}]^{-1/2} \exp \frac{im(x-x')^2}{2(T-T')}.$$

This choice of clock variable for the non-relativistic particle is unique, up to reparameterizations. Any other choice leads to evolving Dirac Observables that cannot be promoted to self-adjoint operators.

For instance if one takes the position as a clock variable: *T=x,* it leads to an evolving observable:

$$X^0(T) = (-q+T)\frac{m}{p} \qquad X^0(T)|_{T=x} = x^0$$

that is not self-adjoint due to the momentum in the denominator.

If the classical Dirac observable can be promoted to a self-adjoint operator in H_{phys} , one can show that there is an operator, *U(T)*, such that the evolution in the c-number parameter T is unitary. The requirement that the evolving observables be self-adjoint is very restrictive in any totally constrained system and imposes strong limitations on the type of clocks that can be used at the quantum level.

The issue of the parameter T

Evolving observables depend on a real parameter T. That is we are assuming that there is an external quantity T , that is not represent by any quantum operator nor belongs to any physical Hilbert space.
One may wonder about the meaning of the condition $q^0 = T$ in the generic situation in which the clock variable q^0 is not defined in H_{phys}

$$q^0 \,|\psi>_{ph} \notin H_{ph}$$

In any generally covariant system as general relativity the clock will be associated to certain physical sub-system with dynamical variables that will not be well defined in H_{phys} . We don't have any external variable.

Evolving constants are measurable quantities but, in the quantum realm, they depend on an external parameter, whose observation is not described by the theory.

Marolf in a very interesting paper has recently presented an implementation of the evolving Dirac observables. We consider however that the issue of the external parameter is still present in this implementation. **Phys.Rev.D79:084016,2009** gr-qc/0902.155

2) Conditional probabilities.

The second alternative we want to consider is a description of the evolution in terms of conditional probabilities.

The idea is that one promotes all variables to quantum operators and computes conditional probabilities among them. This idea appears simple, natural and attractive in a closed system.

Unfortunately one runs into problems due to the totally constrained nature of gravity. Which variables to promote? Dirac observables? Page and Wootters proposed using kinematical variables, not Dirac observables. That way they had some form of evolution. **Phys.Rev.D27:2885,(1983)**

Kuchař in his review on the problem of time, noted that this procedure faces important difficulties, in particular it does not lead to the correct propagators in model systems. The root of the problem is the distributional nature of physical states and the attempt to compute expectation values of kinematical quantities with them.

Other attempts:

A few years ago the idea also received attention by *Dolby* who proposed a new approach to the issue of the definition of conditional probabilities. **gr-qc/0406034**

Hellmann, Mondragon Perez and Rovelli **Phys.Rev.D75:084033,(2007)** analyzed the issue of the definition of probabilities for sequences of measurements proposed by Dolby and concluded that they present interpretational problems, and that it is not clear to what measurement setup does these probabilities correspond.

Very recently, *Brunetti, Fredenhagen and Hoge* have formalized the Page and Wootters construction, in particular the use of distributional states in the kinematical space. At the moment it is unclear if this solves the propagator issue. **arXiv:0909.1899**

3) Conditional probabilities in terms of evolving Dirac observables.

As we have seen, both approaches require the use of variables which are not defined in the physical space.

Here we will elaborate upon a different approach where all reference to external parameters is abolished, and evolving constants are used to define correlations between Dirac observables in the theory.

We propose to revisit the Page-Wootters construction by **computing relational probabilities among evolving Dirac observables.** The latter are well defined on the physical space of states of the theory and are quantities that one can expect to observe and to be represented by well defined self-adjoint quantum operators.

First you choose an evolving observable as your clock, let us call it *T(t) .*
Then one identifies the set of observables $O_1(t)\ldots O_N(t)$ that commute with T and describes the physical system whose evolution one wants to study and computes

$$\mathcal{P}\left(O \in [O_0 - \Delta O, O_0 + \Delta O] | T \in [T_0 - \Delta T, T_0 + \Delta T]\right) = \lim_{\tau \to \infty} \frac{\int_{-\tau}^{\tau} dt\, \mathrm{Tr}\left(P_{O_0}(t) P_{T_0}(t) \rho P_{T_0}(t)\right)}{\int_{-\tau}^{\tau} dt\, \mathrm{Tr}\left(P_{T_0}(t) \rho\right)}$$

Notice that we have changed the notation and the external parameter is now called *t.*

In other words, *t* is the parameter associated to the variable used to define the evolving observables. This variable is treated as an ideal quantity that we do not need to observe.

What does such a probability represent?

The experimental setup we have in mind is to consider an ensamble of non-interacting systems with two quantum variables each to be measured. Each systems is equipped with a recording device that takes a single snapshot of O and T at a "random" unknown value of the "ideal" time *t*. One takes a large number of such systems, launches them all in the same quantum state, "waits for a long time" and concludes the experiment.

From here one can immediately compute frequencies and the joint probability in the limit of infinite systems.

A simple example.

One considers the constrained system:

$$\phi = p_0 + H(q^a, p_a) = 0 \qquad H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m}$$

We have two free particles and one can define:

$$X_1(t) = q^1 - \frac{p_1}{m} q^0 + \frac{p_1}{m} t \qquad X_1(t)\,|_{t=q^0} = q^1$$

$$X_2(t) = q^2 - \frac{p_2}{m} q^0 + \frac{p_2}{m} t$$

We are using q^0 as unobservable parameter and compute $P(X_2 \mid X_1)$

We can then write the conditional probabilities that yield the propagators,

$$P(X_2^f \mid T_2 = X_1^f, X_2^i, T_1 = X_1^i, \rho) \equiv$$

$$\lim_{\tau\to\infty} \frac{\int_{-\tau}^{\tau} dt \int_{-\tau}^{\tau} dt' Tr(P_{X_2^f, X_1^f}(t) P_{X_2^i, X_1^i}(t') \rho P_{X_2^i, X_1^i}(t'))}{\int_{-\tau}^{\tau} dt \int_{-\tau}^{\tau} dt' Tr(P_{X_1^f}(t) P_{X_2^i, X_1^i}(t') \rho P_{X_2^i, X_1^i}(t'))}$$

This expression yields the propagator for the system to move from

$$X_2^i, X_1^i \qquad \text{to} \qquad X_2^f, X_1^f$$

Notice that in particular no assumption about the relative ordering of the unobservable variables *t* and *t'* is needed.

One can show that this expression yields the correct propagator. In the example of the previous slide:

$$P\left(x_2'|x_1', x_2, x_1, \rho_0\right) \sim$$
$$\lim_{\tau\to\infty} \int_0^{\tau} dt' |\langle x_2', t'|x_2, t(x_1)\rangle|^2 \mathcal{P}_{x_1'}(t')\Delta x_2$$

Everything is given in terms of the Dirac Observables $X_1(t), X_2(t)$

Real clocks and loss of unitarity.

Let us come back to the previous result

$$P\left(x_2'|x_1', x_2, x_1, \rho_0\right) \sim \lim_{\tau\to\infty}\int_0^\tau dt' |\langle x_2', t'|x_2, t(x_1)\rangle|^2 \mathcal{P}_{x_1'}(t')\Delta x_2$$

$$\mathcal{P}_{x_1'}(t') \equiv \mathrm{Tr}\left(P_{x_1'}(t')\rho_0\right) / \int_{-\infty}^{\infty} dt\, \mathrm{Tr}\left(P_{x_1'}(t)\rho_0\right)$$

$$\int \mathcal{P}_{x'_1}(t')dt' = 1$$

And $\mathcal{P}_{x'_1}(t')$ can be interpreted as the probability that the external unobservable time q^0 is t' when the variable taken as a clock reads x'_1

This probability will be controlled by the position of the peak and the width of the wave packet of the particle 1. If $\mathcal{P}_{x'_1}(t')$ were a Dirac delta we would recover the exact ordinary non-relativistic propagator.

The use of real clocks may lead to a loss of quantum coherence and therefore to corrections to the standard propagator.

$$P(x'_2 | x'_1, x_2, x_1) = \int Tr[\rho^H_{x_2 x_1} P^H_{x'_2}(t')]\mathcal{P}_{x'_1}(t')dt' = \int Tr[\rho^S_{x_2 x_1}(t')P^S_{x'_2}]\mathcal{P}_{x'_1}(t')dt' = Tr[\rho_{x_2 x_1}(x'_1)P_{x'_2}]$$

$$\rho(T = x'_1) = \int dt' \mathcal{P}_{x'_1}(t')U(t', t(x_1))\rho_{x_2 x_1}U^+(t', t(x_1))$$

We have therefore ended with the standard probability expression with an "effective" density matrix in the Schrödinger picture given by $\rho(T)$
Unitarity may be lost since one ends up with a density matrix that is a superposition of density matrices associated with different values of *t*

The underlying unitary evolution of the evolving constants in the ideal time *t* is crucial, yet unobservable. All we observe are the correlations in physical time, then it is not surprising that they present a fundamental level of loss of coherence due to the Intrinsically limitations of real clocks.

If we assume the "real clock" is behaving semi-classically.

$$\mathcal{P}_t(T) = f(T-t) = \delta(T-t) + a(T)\delta'(T-t) + b(T)\delta''(T-t) + \ldots$$

The Schrödinger evolution is modified: RG, R. Porto, JP, NJP 6, 45 (2004)

$$-i\hbar\frac{\partial\rho}{\partial T} = [\hat{H},\rho] + \sigma(T)[\hat{H},[\hat{H},\rho]] + \ldots \qquad \sigma(T) = \partial b(T)/\partial T.$$

If we assume σ is constant, the equation can be solved exactly and one gets that the density matrix in an energy eigen-basis evolves as

$$\rho_{2\,nm}(t) = \rho_{2\,nm}(0)e^{-i\omega_{nm}t}e^{(-\sigma(\omega_{nm})^2)t} \qquad \omega_{mn} = E_m - E_n$$

Therefore, the off-diagonal elements of the density matrix decay to zero exponentially, and pure states generically evolve into mixed states. Quantum mechanics with real clocks therefore does not have a unitary evolution.

The effects are more pronounced the worse the clock is. Which raises the question: is there a fundamental limitation to how good a clock can be?

There are many phenomenological arguments based on quantum and gravitational considerations that lead to estimates of such a limitation, (Salecker-Wigner and Ng, Karolyhazy, Lloyd, Hogan, Amelino Camelia) $\delta T = T^{1/3} t_P^{2/3}$

We will not enter into the analysis of these phenomenological estimations, (which have been questioned in the literature. But it is important to remark that the evolution with real clocks will not be unitary if the spread in the error of the clock grows with time with some power of *T*.

That is, if $\delta T = T_{planck}$ he evolution is unitary, but if $\delta T = T^a T_{Planck}^{1-a}$ with a>0 there will Exist a fundamental loss of unitarity.

Conclusions:

- Using evolving constants of the motion in the conditional probability interpretation of Page and Wootters allows to correctly compute the propagator and assign probabilities to histories.
- The resulting description is entirely in terms of Dirac observables.
- There are corrections to the propagator due to the use of "real clocks and rods" to measure space and time.

Happy 50th ADM. You and your ideas look as young and vibrant as ever!

Bornholm 1959

From the left, Richard Arnowitt, Charles Misner and Stanley Deser