Black Hole Entropy: An ADM Approach

Steve Carlip

U.C. Davis

ADM-50
College Station, Texas
November 2009

Black holes behave as thermodynamic objects

$$\boldsymbol{T = \frac{\hbar\kappa}{2\pi c}}$$

$$\boldsymbol{S_{BH} = \frac{A}{4\hbar G}}$$

Quantum ($\hbar$) and gravitational ($\boldsymbol{G}$)

Does this thermodynamic behavior have a microscopic explanation?

The problem of “universality”

Black hole entropy counts:

- Weakly coupled string and D-brane states
- Horizonless “fuzzball” geometries
- States in a dual conformal field theory “at infinity”
- Spin network states crossing the horizon
- Spin network states inside the horizon
- “Heavy” degrees of freedom in induced gravity
- Entanglement entropy (maybe holographic)
- Points in a causal set
  in the horizon’s domain of dependence
- States of a conformal field theory near an extremal horizon
- No local states—it’s inherently global
- Nothing—it comes from quantum field theory in a fixed
  background, and doesn’t know about quantum gravity

Answer: apparently, all of the above

Is there an underlying mechanism that can
explain why these approaches all agree?

A small detour: entropy and the Cardy formula

Any two-dimensional conformal field theory can be characterized by generators $L[\xi]$, $\bar{L}[\bar{\xi}]$ of holomorphic and antiholomorphic diffeomorphisms

Virasoro algebra:

$$[L[\xi], L[\eta]] = L[\eta\xi' - \xi\eta'] + \frac{c}{48\pi}\int dz\,(\eta'\xi'' - \xi'\eta'')$$

Central charge c (“conformal anomaly”) depends on theory
Conserved charge $L_0 \sim$ energy

Cardy: the density of states at temperature T is asymptotically

$$\ln \rho(L_0) \sim \frac{\pi^2}{3}cT$$

Entropy is fixed by symmetry, independent of details!

Why this might help:
matter near a horizon looks conformal

Black hole in "tortoise" coordinates:

$$ds^2 = N^2(dt^2 - dr_*{}^2) + ds_\perp{}^2$$

($N \longrightarrow 0$ at horizon)

Scalar field:

$$(\Box - m^2)\varphi = \frac{1}{N^2}(\partial_t^2 - \partial_{r_*}^2)\varphi + O(1)$$

Mass and transverse excitations become negligible
Effective two-dimensional conformal field (at each point)

Wilczek, Robinson, Iso, Morita, Umetsu:
two-dimensional CFT gives Hawking flux, spectrum

Medved, Martin, Visser:
conformal symmetry is generic at Killing horizon

An ADM Approach

Basic philosophy:

– Conditional probability: must impose presence of a black hole
– Need “boundary conditions” at horizon
– Diffeomorphisms must respect boundary conditions:
some generators become symmetries rather than invariances
– New Goldstone-like degrees of freedom

More specifically:

– Conformal anomaly breaks diffeomorphism invariance

$$L[\xi]|phys\rangle = \bar{L}[\xi]|phys\rangle = 0$$

Not consistent with Virasoro algebra with $c \neq 0$:

Must weaken constraints— e.g., only require positive-frequency part annihilate $|phys\rangle$
$\Rightarrow$ formerly nonphysical “gauge” states become physical

Strategy:

Brown and Henneaux:
look at boundary term in ADM Hamiltonian

$$\delta H[\xi] = \textit{bulk terms} + \int_{\partial\Sigma} \Theta[\delta g]$$
$$\delta B[\xi] = - \int_{\partial\Sigma} \Theta[\delta g]$$

Full Hamiltonian $(H + B)[\xi]$ generates diffeos/surface deformations, so have Dirac brackets

$$\{(H + B)[\xi], (H + B)[\eta])\} = \delta_\xi (H + B)[\eta] \approx \delta_\xi B[\eta]$$
$$\Rightarrow \{B[\xi], B[\eta])\}^* = - \int_{\partial\Sigma} \Theta[\delta g]$$

General structure:

$$\{B[\xi], B[\eta])\}^* = B[\{\xi, \eta\}] + K[\xi, \eta]$$

Implementation:

Metric

$$ds^2 = -N^2 dt^2 + d\rho^2 + \sigma_{\alpha\beta} dx^\alpha dx^\beta$$

Surface gravity

$$\kappa = n^a \partial_a N$$

Stretched horizon $N = \epsilon$; restrict diffeomorphisms to

$$\delta_\xi N = 0 \Rightarrow \xi^\rho = -\rho \partial_t \xi^t = -\frac{1}{\kappa} \partial_t \xi^\perp$$

$$\delta g_{\rho\alpha} = 0 \Rightarrow \partial_\rho \xi^\alpha = -\sigma^{\alpha\beta} \partial_\beta \xi^\rho$$

At stretched horizon, ADM Hamiltonian has a boundary term:

$$\delta H[\xi] = \cdots - \frac{1}{16\pi G}\int_{\partial\Sigma}\Big[\sqrt{\sigma}\left(n^c g^{bd} - n^b g^{cd}\right)\left(\xi^{\perp}\nabla_b \delta g_{cd} - \nabla_b \xi^{\perp}\delta g_{cd}\right) + 2\xi^a \delta\pi_a{}^{\rho} - \xi^{\rho}\pi^{ab}\delta g_{ab}\Big]$$

Evaluate for a surface deformation η:

$$\delta_\eta H[\xi] = \cdots - \frac{1}{8\pi G}\int_{\partial\Sigma}\Big[\xi^{\perp}\sigma^{ab}\nabla_a\nabla_b(n_c\eta^c) + n^a\partial_a\xi^{\perp}\,\sigma^{bc}\nabla_b\eta_c - (\xi\leftrightarrow\eta)\Big]\sqrt{\sigma}$$

$$= \cdots - \frac{1}{16\pi G\kappa}\int_{\partial\Sigma}\Big[\partial_t\xi^{\perp}\Delta_{\Sigma}\eta^{\perp} - \partial_t\eta^{\perp}\Delta_{\Sigma}\xi^{\perp}\Big]\sqrt{\sigma}$$

Need one more relation between ∂_α and ∂_t:
possibly from existence of Hamiltonian

$$\Delta_\Sigma \xi^\perp - \frac{1}{N^2}{\partial_t}^2 \xi^\perp = 0$$

Then have central term

$$\frac{1}{16\pi G\kappa}\int_{\partial\Sigma}\left[\partial_t \xi^t {\partial_t}^2 \eta^t - \partial_t \eta^t {\partial_t}^2 \xi^t\right]\sqrt{\sigma} \Rightarrow c = \frac{3A}{2\pi G\kappa}$$

Cardy formula then gives

$$S = \frac{\pi^2}{3} cT = \frac{\pi^2}{3}\cdot\frac{3A}{2\pi G\kappa}\cdot\frac{\kappa}{2\pi} = \frac{A}{4G}$$

Related approaches:

– Particular conformal algebras for BTZ black hole, extremal Kerr black hole
– Explicit construction of boundary dynamics in 2+1 dimensions
– Covariant phase space methods
– Near-horizon conformal symmetries
– Adding horizon condition as genuine constraint

Universality?

Does this symmetry show up in other (“unrelated”) approaches?

– AdS/CFT: related to near-horizon CFT
– Loop quantum gravity: interesting coincidences in central charges. . .
– Fuzzballs: under investigation
– Path integral? (Cardy formula as measure?)