Lagrangians and Hamiltonians in Classical Field Theory

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Lagrangians and Hamiltonians in Classical Field Theory

Lagrangian and Hamiltonian formulations of field theories play a central role in their quantization. However, it had been my view that their role in classical field theory was not much more than that of a mnemonic device to remember the field equations. When I wrote my GR text, the discussion of the Lagrangian (Einstein-Hilbert) and Hamiltonian (ADM) formulations of general relativity was relegated to an appendix. My views have changed dramatically in the past 15 years: The existence of a Lagrangian or Hamiltonian provides important auxiliary structure to a classical field theory, which endows the theory with key properties.

Lagrangians and Hamiltonians in Particle Mechanics

Consider particle paths $q(t)$. If L is a function of $(q, \dot{q})$, then we have the identity

$$\delta L = [\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}]\delta q + \frac{d}{dt}[\frac{\partial L}{\partial \dot{q}}\delta q]$$

holding at each time t. L is a Lagrangian for the system if the equations of motion are

$$0 = E \equiv \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

The "boundary term"

$$\Theta(q, \dot{q}) \equiv \frac{\partial L}{\partial \dot{q}}\delta q = p\delta q$$

(with $p \equiv \partial L/\partial \dot{q}$) is usually discarded. However, by taking a second, antisymmetrized variation of Θ and evaluating at time t_0, we obtain the quantity

$$\begin{aligned}\Omega(q, \delta_1 q, \delta_2 q) &= [\delta_1 \Theta(q, \delta_2 q) - \delta_2 \Theta(q, \delta_1 q)]|_{t_0} \\ &= [\delta_1 p \delta_2 q - \delta_2 p \delta_1 q]|_{t_0}\end{aligned}$$

Then Ω is independent of t_0 provided that the varied paths $\delta_1 q(t)$ and $\delta_2 q(t)$ satisfy the linearized equations of motion about $q(t)$. Ω is highly degenerate on the infinite dimensional space of all paths $\mathcal{F}$, but if we factor $\mathcal{F}$ by the degeneracy subspaces of Ω, we obtain a finite dimensional *phase space* Γ on which Ω is non-degenerate. A *Hamiltonian*, H, is a function on Γ whose pullback to

$\mathcal{F}$ satisfies

$$\delta H = \Omega(q; \delta q, \dot{q})$$

for all δq provided that $q(t)$ satisfies the equations of motion. This is equivalent to saying that the equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

Lagrangians and Hamiltonians in Classical Field Theory

Let ϕ denote the collection of dynamical fields. The analog of $\mathcal{F}$ is the space of field configurations on spacetime. For an n-dimensional spacetime, a Lagrangian $\mathbf{L}$ is most naturally viewed as an n-form on spacetime that is a function of ϕ and finitely many of its derivatives. Variation of $\mathbf{L}$ yields

$$\delta \mathbf{L} = \mathbf{E}\delta\phi + d\mathbf{\Theta}$$

where $\mathbf{\Theta}$ is an $(n-1)$-form on spacetime, locally constructed from ϕ and $\delta\phi$. The equations of motion are then $\mathbf{E} = 0$. The symplectic current $\boldsymbol{\omega}$ is defined by

$$\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\boldsymbol{\Theta}(\phi, \delta_2\phi) - \delta_2\boldsymbol{\Theta}(\phi, \delta_1\phi)$$

and Ω is then defined by

$$\Omega(\phi, \delta_1\phi, \delta_2\phi) = \int_\Sigma \boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi)$$

where Σ is a Cauchy surface. Phase space is constructed by factoring field configuration space by the degeneracy subspaces of Ω, and a Hamiltonian, H_ξ, conjugate to a vector field ξ^a on spacetime is a function on phase space whose pullback to field configuration space satisfies

$$\delta H_\xi = \Omega(\phi; \delta\phi, \mathcal{L}_\xi\phi)$$

Diffeomorphism Covariant Theories

A diffeomorphism covariant theory is one whose Lagrangian is constructed entirely from dynamical fields, i.e., there is no "background structure" in the theory apart from the manifold structure of spacetime. For a diffeomorphism covariant theory for which dynamical fields, ϕ, are a metric g_{ab} and tensor fields ψ, the Lagrangian takes the form

$$\mathbf{L} = \mathbf{L}\left(g_{ab}, R_{bcde}, ..., \nabla_{(a_1} ... \nabla_{a_m)} R_{bcde}; \psi, ..., \nabla_{(a_1} ... \nabla_{a_l)} \psi\right)$$

Noether Current and Noether Charge

For a diffeomorphism covariant theory, every vector field ξ^a on spacetime generates a local symmetry. We associate to each ξ^a and each field configuration, ϕ (*not* required, at this stage, to be a solution of the equations of motion), a Noether current $(n-1)$-form, $\mathbf{J}_\xi$, defined by

$$\mathbf{J}_\xi = \mathbf{\Theta}(\phi, \mathcal{L}_\xi \phi) - \xi \cdot \mathbf{L}$$

A simple calculation yields

$$d\mathbf{J}_\xi = -\mathbf{E}\mathcal{L}_\xi \phi$$

which shows $\mathbf{J}_\xi$ is closed (for all ξ^a) when the equations of motion are satisfied. It can then be shown that for all

ξ^a and all ϕ (not required to be a solution to the equations of motion), we can write $\mathbf{J}_\xi$ as

$$\mathbf{J}_\xi = \xi^a \mathbf{C}_a + d\mathbf{Q}_\xi$$

where $\mathbf{C}_a = 0$ are the constraint equations of the theory and $\mathbf{Q}_\xi$ is an $(n-2)$-form locally constructed out of the dynamical fields ϕ, the vector field ξ^a, and finitely many of their derivatives. It can be shown that $\mathbf{Q}_\xi$ can always be expressed in the form

$$\mathbf{Q}_\xi = \mathbf{W}_c(\phi)\xi^c + \mathbf{X}^{cd}(\phi)\nabla_{[c}\xi_{d]} + \mathbf{Y}(\phi, \mathcal{L}_\xi\phi) + d\mathbf{Z}(\phi, \xi)$$

Furthermore, there is some "gauge freedom" in the choice of $\mathbf{Q}_\xi$ arising from (i) the freedom to add an exact form to the Lagrangian, (ii) the freedom to add an exact form to $\mathbf{\Theta}$, and (iii) the freedom to add an exact form to $\mathbf{Q}_\xi$. Using this freedom, we may choose $\mathbf{Q}_\xi$ to take the form

$$\mathbf{Q}_\xi = \mathbf{W}_c(\phi)\xi^c + \mathbf{X}^{cd}(\phi)\nabla_{[c}\xi_{d]}$$

where

$$(\mathbf{X}^{cd})_{c_3 \ldots c_n} = -E_R^{abcd}\boldsymbol{\epsilon}_{abc_3 \ldots c_n}$$

where $E_R^{abcd} = 0$ are the equations of motion that would result from pretending that R_{abcd} were an independent dynamical field in the Lagrangian $\mathbf{L}$.

Hamiltonians

Let ϕ be any solution of the equations of motion, and let $\delta\phi$ be any variation of the dynamical fields (not necessarily satisfying the linearized equations of motion) about ϕ. Let ξ^a be an arbitrary, fixed vector field. We then have

$$\begin{aligned}
\delta\mathbf{J}_\xi &= \delta\mathbf{\Theta}(\phi, \mathcal{L}_\xi\phi) - \xi\cdot\delta\mathbf{L} \\
&= \delta\mathbf{\Theta}(\phi, \mathcal{L}_\xi\phi) - \xi\cdot d\mathbf{\Theta}(\phi, \delta\phi) \\
&= \delta\mathbf{\Theta}(\phi, \mathcal{L}_\xi\phi) - \mathcal{L}_\xi\mathbf{\Theta}(\phi, \delta\phi) + d(\xi\cdot\mathbf{\Theta}(\phi, \delta\phi))
\end{aligned}$$

On the other hand, we have

$$\delta\mathbf{\Theta}(\phi, \mathcal{L}_\xi\phi) - \mathcal{L}_\xi\mathbf{\Theta}(\phi, \delta\phi) = \boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi\phi)$$

We therefore obtain

$$\boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi \phi) = \delta \mathbf{J}_\xi - d(\xi \cdot \boldsymbol{\Theta})$$

Replacing $\mathbf{J}_\xi$ by $\xi^a \mathbf{C}_a + d\mathbf{Q}_\xi$ and integrating over a Cauchy surface Σ, we obtain

$$\begin{aligned} \Omega(\phi, \delta\phi, \mathcal{L}_\xi \phi) &= \int_\Sigma [\xi^a \delta \mathbf{C}_a + \delta d\mathbf{Q}_\xi - d(\xi \cdot \boldsymbol{\Theta})] \\ &= \int_\Sigma \xi^a \delta \mathbf{C}_a + \int_{\partial\Sigma} [\delta Q_\xi - \xi \cdot \boldsymbol{\Theta})] \end{aligned}$$

The $(n-1)$-form $\boldsymbol{\Theta}$ cannot be written as the variation of a quantity locally and covariantly constructed out of the dynamical fields (unless $\boldsymbol{\omega} = 0$). However, it is possible that for the class of spacetimes being considered, we can find a (not necessarily diffeomorphism covariant) $(n-1)$-form, $\mathbf{B}$, such that

$$\delta \int_{\partial\Sigma} \xi \cdot \mathbf{B} = \int_{\partial\Sigma} \xi \cdot \mathbf{\Theta}$$

A Hamiltonian for the dynamics generated by ξ^a exist on this class of spacetimes if and only if such a $\mathbf{B}$ exists. This Hamiltonian is then given by

$$H_\xi = \int_\Sigma \xi^a \mathbf{C}_a + \int_{\partial\Sigma} [\mathbf{Q}_\xi - \xi \cdot \mathbf{B}]$$

Note that "on shell", i.e., when the field equations are satisfied, we have $\mathbf{C}_a = 0$ so the Hamiltonian is purely a "surface term".

Energy and Angular Momentum

If a Hamiltonian conjugate to a time translation $\xi^a = t^a$ exists, we define the *energy*, $\mathcal{E}$ of a solution $\phi = (g_{ab}, \psi)$ by

$$\mathcal{E} \equiv H_t = \int_{\partial\Sigma} (\mathbf{Q}_t - t \cdot \mathbf{B})$$

Similarly, if a Hamiltonian, H_φ, conjugate to a rotation $\xi^a = \varphi^a$ exists, we define the *angular momentum*, $\mathcal{J}$ of a solution by

$$\mathcal{J} \equiv -H_\varphi = -\int_{\partial\Sigma} [\mathbf{Q}_\varphi - \varphi \cdot \mathbf{B}]$$

If φ^a is tangent to Σ, the last term vanishes, and we obtain simply

$$\mathcal{J} = -\int_{\partial\Sigma} \mathbf{Q}_{\varphi}$$

Energy and Angular Momentum in General Relativity:

ADM vs Komar

In general relativity in 4 dimensions, the Einstein-Hilbert Lagrangian is

$$\mathbf{L}_{abcd} = \frac{1}{16\pi}\boldsymbol{\epsilon}_{abcd}R$$

This yields the symplectic potential 3-form

$$\mathbf{\Theta}_{abc} = \boldsymbol{\epsilon}_{dabc}\frac{1}{16\pi}g^{de}g^{fh}\left(\nabla_f \delta g_{eh} - \nabla_e \delta g_{fh}\right).$$

The corresponding Noether current and Noether charge are

$$(\mathbf{J}_\xi)_{abc} = \frac{1}{8\pi}\boldsymbol{\epsilon}_{dabc}\nabla_e\left(\nabla^{[e}\xi^{d]}\right),$$

and

$$(\mathbf{Q}_\xi)_{ab} = -\frac{1}{16\pi}\boldsymbol{\epsilon}_{abcd}\nabla^c\xi^d.$$

For asymptotically flat spacetimes, the formula for angular momentum conjugate to an asymptotic rotation φ^a is

$$\mathcal{J} = \frac{1}{16\pi}\int_\infty \boldsymbol{\epsilon}_{abcd}\nabla^c\varphi^d$$

This agrees with the ADM expression, and when φ^a is a Killing vector field, it agrees with the Komar formula. For an asymptotic time translation t^a, a Hamiltonian, H_t, exists with

$$t^a\mathbf{B}_{abc} = -\frac{1}{16\pi}\tilde{\boldsymbol{\epsilon}}_{bc}\left((\partial_r g_{tt} - \partial_t g_{rt}) + r^k h^{ij}(\partial_i h_{kj} - \partial_k h_{ij})\right)$$

The corresponding Hamiltonian

$$H_t = \int_\Sigma t^a \mathbf{C}_a + \frac{1}{16\pi} \int_\infty dS r^k h^{ij} (\partial_i h_{kj} - \partial_k h_{ij})$$

is precisely the ADM Hamiltonian, and the surface term is the ADM mass,

$$M_{\mathrm{ADM}} = \frac{1}{16\pi} \int_\infty dS r^k h^{ij} (\partial_i h_{kj} - \partial_k h_{ij})$$

By contrast, if t^a is a Killing field, the Komar expression

$$M_{\mathrm{Komar}} = -\frac{1}{8\pi} \int_\infty \boldsymbol{\epsilon}_{abcd} \nabla^c t^d$$

happens to give the correct (ADM) answer, but this is merely a fluke.

The First Law of Black Hole Mechanics

Return to a general, diffeomorphism covariant theory, and recall that for any solution ϕ, any $\delta\phi$ (not necessarily a solution of the linearized equations) and any ξ^a, we have

$$\Omega(\phi, \delta\phi, \mathcal{L}_\xi \phi) = \int_\Sigma \xi^a \delta \mathbf{C}_a + \int_{\partial\Sigma} [\delta Q_\xi - \xi \cdot \mathbf{\Theta})]$$

Now suppose that ϕ is a stationary black hole with a Killing horizon with bifurcation surface $\mathcal{H}$. Let ξ^a denote the horizon Killing field, so that $\xi^a|_\mathcal{H} = 0$ and

$$\xi^a = t^a + \Omega_H \varphi^a$$

Then $\mathcal{L}_\xi \phi = 0$. Let $\delta\phi$ satisfy the linearized equations, so $\delta \mathbf{C}_a = 0$. Let Σ be a hypersurface extending from $\mathcal{H}$

to infinity.

$$0 = \int_\infty [\delta Q_\xi - \xi \cdot \mathbf{\Theta})] - \int_\mathcal{H} \delta Q_\xi$$

Thus, we obtain

$$\delta \int_\mathcal{H} Q_\xi = \delta\mathcal{E} - \Omega_H \delta\mathcal{J}$$

Furthermore, from the formula for Q_ξ and the properties of Killing horizons, one can show that

$$\delta \int_\mathcal{H} Q_\xi = \frac{\kappa}{2\pi}\delta S$$

where S is defined by

$$S = 2\pi \int_{\mathcal{H}} \mathbf{X}^{cd} \epsilon_{cd}$$

where ϵ_{cd} denotes the binormal to $\mathcal{H}$. Thus, we have shown that the first law of black hole mechanics

$$\frac{\kappa}{2\pi} \delta S = \delta \mathcal{E} - \Omega_H \delta \mathcal{J}$$

holds in an arbitrary diffeomorphism covariant theory of gravity, and we have obtained an explicit formula for black hole entropy S.

Variational Principle for Stability

Suppose one is interested in the stability of a stationary solution, ϕ, of the field equations. In certain cases—such as spherically symmetric perturbations of static, spherically solutions of Einstein-fluid equations—it may be possible to fix the gauge and solve the linearized constraint equations in such a way that for the reduced linearized theory, the pullback of the symplectic current to a Cauchy surface Σ takes the form

$$\boldsymbol{\omega} = \mathbf{W}_{\alpha\beta}\left(\psi_1^\alpha \frac{\partial \psi_2^\beta}{\partial t} - \psi_2^\alpha \frac{\partial \psi_1^\beta}{\partial t}\right)$$

where ψ^α denotes the dynamical variables for the reduced linearized theory and $\mathbf{W}_{\alpha\beta}$ is constructed from the quantities appearing in the background solution. It follows that $\int_\Sigma \mathbf{W}_{\alpha\beta}\psi^\alpha\psi^\beta$ is conserved, and, if positive definite, yields an inner product $\langle\, , \,\rangle$. If h is the Hamiltonian for the reduced linearized theory, then

$$I = \frac{h(\psi, \psi)}{\langle \psi, \psi \rangle}$$

provides a variational principle, from which stability can be readily determined.