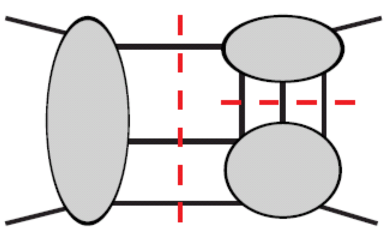
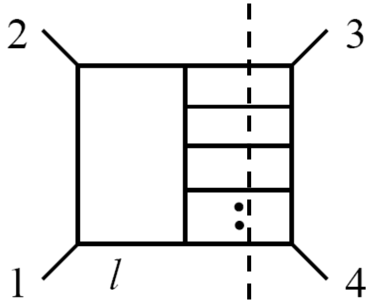
The Ultraviolet Structure of *N* = 8 Supergravity at Four Loops and Beyond

ADM Celebration, Nov 8, 2009

Zvi Bern, UCLA

With: J. J. Carrasco, L. Dixon, H. Johansson, and R. Roiban





Outline

Will present concrete evidence for non-trivial UV cancellations in *N* = 8 supergravity, suggesting it is UV finite.

- **Review of conventional wisdom on UV divergences in quantum gravity.**
- **Remarkable simplicity of gravity scattering amplitudes.**
- **Calculational method:**
  - **(a) Tree-level relations between gravity and gauge theory.**
  - **(b) Unitarity method for loops.**
- **Explicit three-loop calculation.**
- **Explicit four-loop calculation**
- ***All*-loop arguments for UV finiteness of *N* = 8 supergravity.**
- **Origin of cancellation -- generic to all gravity theories.**

N = 8 Supergravity

Eight times the susy of N = 1 theory of Ferrara, Freedman and van Nieuwenhuizen

Consider the N = 8 theory of Cremmer and Julia.

256 massless states

$N=8:$	1	8	28	56	70	56	28	8	1
helicity :	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
	h^-	ψ_i^-	v_{ij}^-	χ_{ijk}^-	s_{ijkl}	χ_{ijk}^+	v_{ij}^+	ψ_i^+	h^+

We calculate at the origin of moduli space with scalars having vanishing vevs. Moduli not visible.

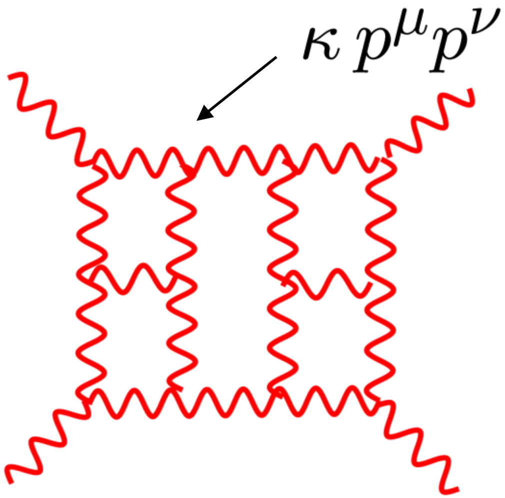
Reasons to focus on this theory:

- **With more susy expect better UV properties.**
- **High symmetry implies technical simplicity.**

Power Counting at High Loop Orders

$\kappa = \sqrt{32\pi G_N}$ ⟵ Dimensionful coupling

See Stelle's and Woodard's talks



Gravity: $$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu) \cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu) \cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV.

Non-renormalizable by power counting.

Much more sophisticated power counting in supersymmetric theories but this is the basic idea.

Divergences in Gravity

One loop: **Vanish on shell**

$R^2, \; R_{\mu\nu}^2, \; R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ **vanishes by Gauss-Bonnet theorem**

Pure gravity 1-loop finite, but *not* with matter

't Hooft, Veltman (1974)
Deser, et al.

Two loop: Pure gravity counterterm has non-zero coefficient:

$$R^3 \equiv R^{\lambda\rho}{}_{\mu\nu}R^{\mu\nu}{}_{\sigma\tau}R^{\sigma\tau}{}_{\lambda\rho}$$

Goroff, Sagnotti (1986); van de Ven (1992)

Any supergravity:

R^3 is ***not*** a valid supersymmetric counterterm.
Produces a helicity amplitude $(-,+,+,+)$ forbidden by susy.

Grisaru (1977); Tomboulis (1977)

The first divergence in *any* supergravity theory can be no earlier than three loops.

R^4 **squared Bel-Robinson tensor expected counterterm**

Deser, Kay, Stelle (1977); Kaku, Townsend, van Nieuwenhuizen (1977), Ferrara, Zumino (1978)

Opinions from the 80's

Unfortunately, in the absence of further mechanisms for cancellation, the analogous $N = 8$ $D = 4$ supergravity theory would seem set to diverge at the three-loop order.

Howe, Stelle (1984)

It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions. … The final word on these issues may have to await further explicit calculations.

Marcus, Sagnotti (1985)

The idea that *all* supergravity theories diverge (at three loops) has been widely accepted for over 25 years

Where is First D= 4 UV Divergence in N = 8 Sugra?

Various opinions over the years:

3 loops	Conventional superspace power counting	Green, Schwarz, Brink (1982) Howe and Stelle (1989) Marcus and Sagnotti (1985)
5 loops	Partial analysis of unitarity cuts; *If* $\mathcal{N}$ = 6 harmonic superspace exists; algebraic renormalisation argument	Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998) Howe and Stelle (2003,2009)
6 loops	*If* $\mathcal{N}$ = 7 harmonic superspace exists	Howe and Stelle (2003)
7 loops	*If* $\mathcal{N}$ = 8 harmonic superspace exists; lightcone gauge locality arguments; Algebraic renormalization arguments	Grisaru and Siegel (1982); Kallosh (2009) Howe, Stelle and Bossard (yesterday)
8 loops	Explicit identification of potential susy invariant counterterm with full non-linear susy	Kallosh; Howe and Lindström (1981)
9 loops	*Assume* Berkovits' superstring non-renormalization theorems can be carried over to D=4 $\mathcal{N}$ = 8 supergravity and extrapolate to 9 loops. Speculations on D=11 gauge invariance.	Green, Russo, Vanhove (2006) Stelle (2006) Berkovits, Green, Russo, Vanhove (2009)

No divergence demonstrated above. Arguments based on lack of susy protection! We will present evidence of all loop finiteness.

To end debate, we need solid results!

Reasons to Reexamine UV Behavior

1) **Discovery of remarkable cancellations at 1 loop – the "no-triangle property".** **Important implication for higher loops!**

ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bjerrum-Bohr, Vanhove; Arkani-Hamed Cachazo, Kaplan; ZB, Dixon, Roiban

2) ***Every* explicit loop calculation to date finds $N = 8$ supergravity has identical power counting as $N = 4$ super-Yang-Mills theory, which is UV finite.** Green, Schwarz and Brink; ZB, Dixon, Dunbar, Perelstein, Rozowsky; Bjerrum-Bohr, Dunbar, Ita, PerkinsRisager; ZB, Carrasco, Dixon, Johanson, Kosower, Roiban.

3) **Interesting hint from string dualities.** Chalmers; Green, Vanhove, Russo

- **Dualities restrict form of effective action. May prevent divergences from appearing in $D = 4$ supergravity, athough indirect nontrivial issues with decoupling of towers of massive states.**

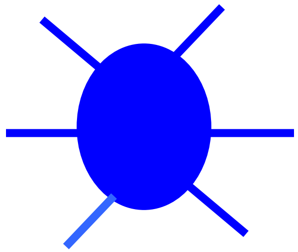
4) **Interesting string non-renormalization theorem from Berkovits. Suggests divergence delayed to nine loops, but needs to be redone directly in field theory, not string theory.** Green, Vanhove, Russo

Off-shell Formalisms

In graduate school you learned that scattering amplitudes need to be calculated using unphysical gauge dependent quantities: Off-shell Green functions and Feynman diagrams.

Standard machinery:

$p^2 \neq m^2$

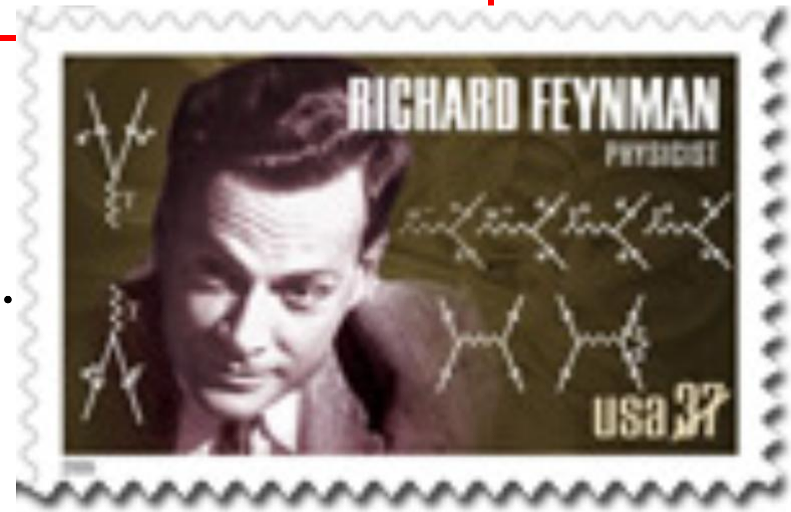


- **Fadeev-Popov procedure for gauge fixing.**
- **Taylor-Slavnov Identities.**
- **BRST.**
- **Gauge fixed Feynman rules.**
- **Batalin-Fradkin-Vilkovisky quantization for gravity.**
- **Off-shell constrained superspaces.**

For all this machinery relatively few calculations in quantum gravity – very few checks of assertions on UV properties.

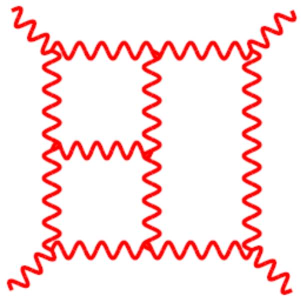
Explicit calculations from 't Hooft and Veltman;
S. Deser et al,
Goroff and Sagnotti; van de Ven

Why are Feynman diagrams clumsy for high-loop processes?



- Vertices and propagators involve gauge-dependent off-shell states. An important origin of the complexity.

$$\int \frac{d^4 p}{(2\pi)^4}$$



$p^2 \neq m^2$

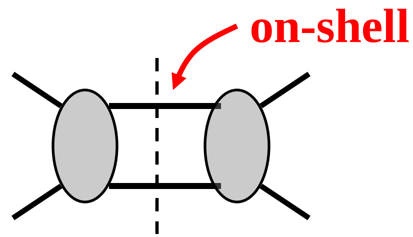
- To get at root cause of the trouble we must rewrite perturbative quantum field theory.

- **All steps should be in terms of gauge invariant on-shell states. On-shell formalism.** $p^2 = m^2$

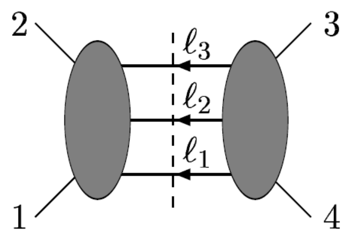
Bern, Dixon, Dunbar and Kosower (BDDK)

Modern Unitarity Method: Rewrite of QFT

Two-particle cut:

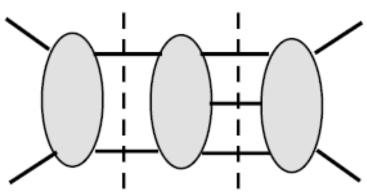


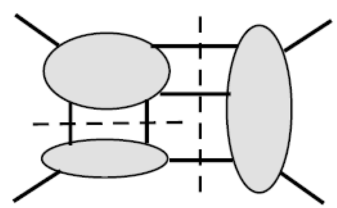
Three-particle cut:



Systematic assembly of complete amplitudes from cuts for any number of particles or loops.

Generalized unitarity as a practical tool:



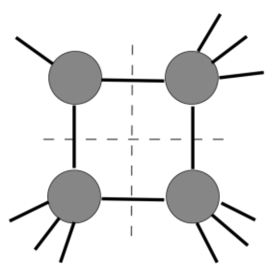


Different cuts merged to give an expression with correct cuts in all channels.

Bern, Dixon and Kosower
Britto, Cachazo and Feng; Forde

complex momenta to solve cuts

Britto, Cachazo and Feng
Buchbinder and Cachazo



Method of Maximal Cuts

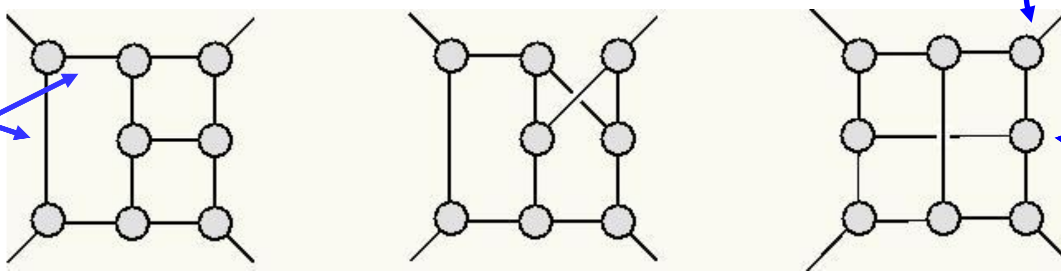
ZB, Carrasco, Johansson, Kosower

A refinement of unitarity method for constructing complete higher-loop amplitudes is “Method of Maximal Cuts”. Systematic construction in any massless theory.

To construct the amplitude we use cuts with maximum number of on-shell propagators:

tree amplitudes

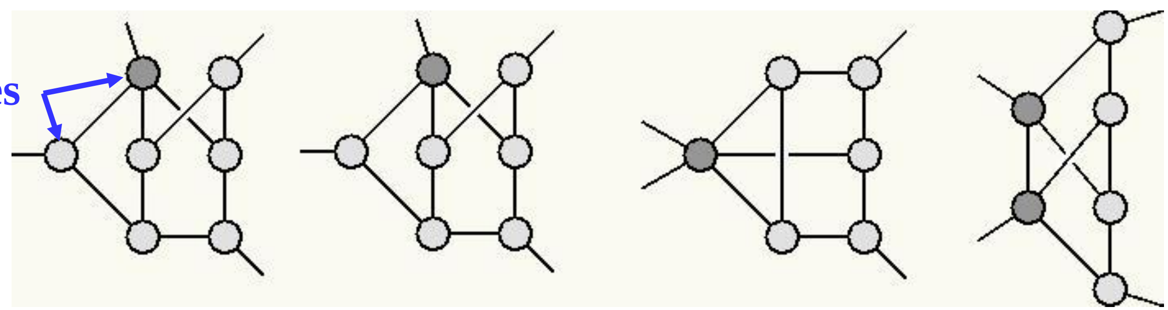
on-shell



Maximum number of propagator placed on-shell.

Then systematically release cut conditions to obtain contact terms:

tree amplitudes



Fewer propagators placed on-shell.

Related to subsequent leading singularity method which uses hidden singularities.

Cachazo and Skinner; Cachazo; Cachazo, Spradlin, Volovich; Spradlin, Volovich, Wen

Gravity vs Gauge Theory

Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R$$

flat metric

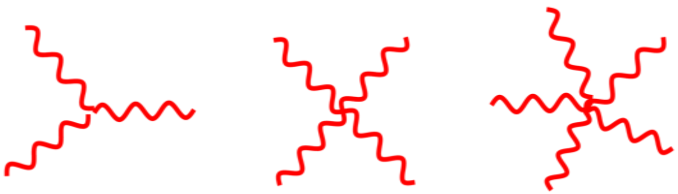
graviton field

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

Infinite number of complicated interactions



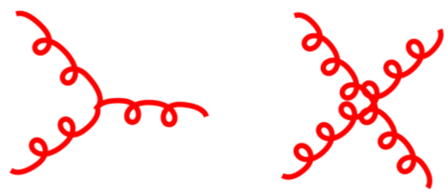
$+ \cdots$

terrible mess

Non-renormalizable

Compare to Yang-Mills Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$

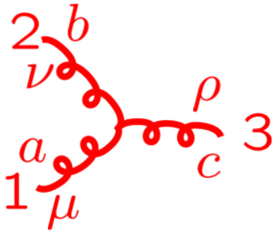


Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Three Vertices

Standard Feynman diagram approach.



Three-gluon vertex:

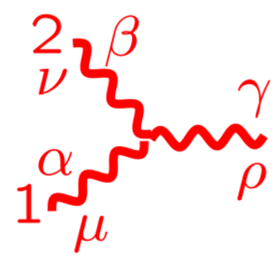
$$V^{abc}_{3\,\mu\nu\sigma} = -g f^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \eta_{\nu\rho}(k_1-k_2)_\mu + \eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

Three-graviton vertex:

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) =$$

$$\begin{aligned}
&\mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\quad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\quad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\quad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.

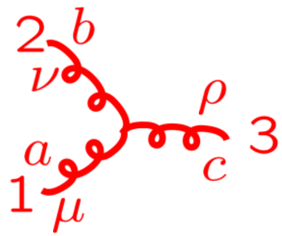
Definitely not a good approach.

Simplicity of Gravity Amplitudes

On-shell **three vertices contain all information:**

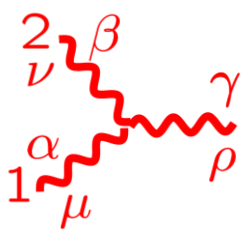
$$k_i^2 = 0$$

gauge theory:



$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$
$$\times\,(\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

"square" of Yang-Mills vertex.

Any gravity scattering amplitude constructible solely from *on-shell* 3 vertex.

- **BCFW on-shell recursion for tree amplitudes.**

Britto, Cachazo, Feng and Witten; Brandhuber, Travaglini, Spence; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

- **Unitarity method for loops.**

ZB, Dixon, Dunbar and Kosower; ZB, Dixon, Kosower; Britto, Cachazo, Feng; ZB, Morgan; Buchbinder and Cachazo; ZB, Carrasco, Johansson, Kosower; Cachzo and Skinner.

Gravity vs Gauge Theory

Consider the gravity Lagrangian

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flat metric

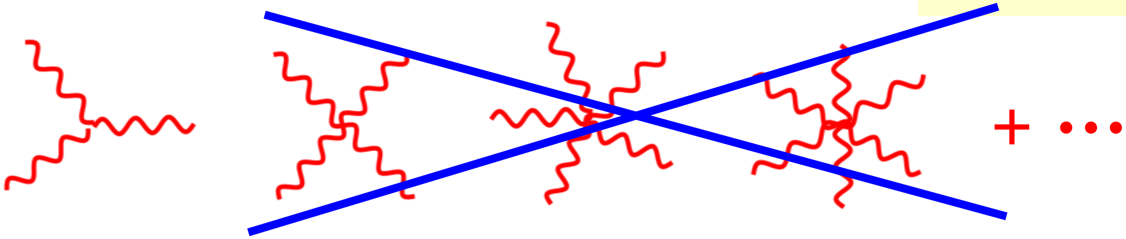
graviton field

metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

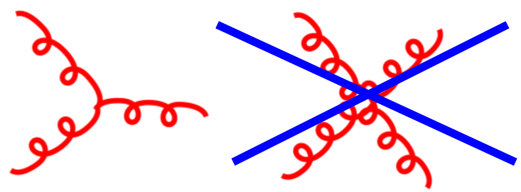
Infinite number of irrelevant interactions!



Simple relation to gauge theory

Compare to Yang-Mills Lagrangian

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$



Only three-point interactions

Gravity seems ~~so much~~ no more complicated than gauge theory.

KLT Relations Between Gravity and Gauge Theory

At *tree level* Kawai, Lewellen and Tye derived a relationship between closed and open string amplitudes.
In field theory limit, relationship is between gravity and gauge theory.

$$M_4^{\rm tree}(1,2,3,4) = s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = s_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) + s_{13}s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

Gravity amplitude

Color stripped gauge theory amplitude

where we have stripped all coupling constants

$$\mathcal{A}_4^{\rm tree} = g^2 \sum_{\rm non-cyclic} {\rm Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_4^{\rm tree}(1,2,3,4)$$

Full gauge theory amplitude

Holds for any external states.
See review: gr-qc/0206071

Progress in gauge theory can be imported into gravity theories

Higher-Point Gravity and Gauge Theory

ZB, Carrasco, Johansson

We recently found a much simpler relation between gravity and gauge theory.

color factor

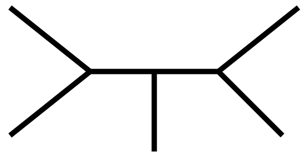
kinematic numerator factor

Gauge theory: $$\mathcal{A}_5^{\text{tree}} = i g^{n-2} \sum_{i=1}^{15} \frac{c_i\, n_i}{D_i}$$

Feynman propagators

sum over diagrams

Einstein Gravity: $$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$



- **Same relations between *N* = 4 sYM and *N* = 8 sugra – in fact discovered first in these theories from studying 4 loops!**
- **An important tool for high-loop *N*=8 supergravity.**

The key to gravity is to map it into two copies of gauge theory

N = 8 Supergravity

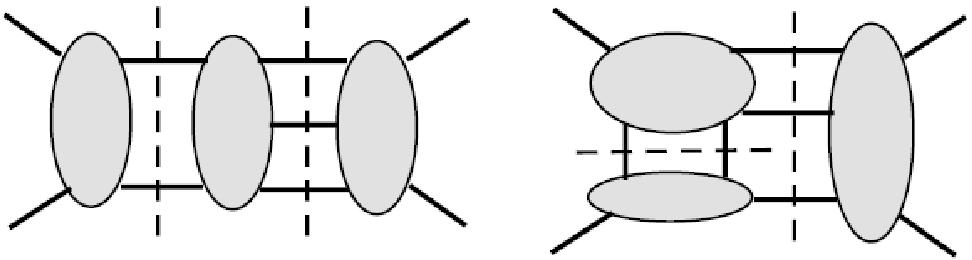
Key trick is to use cuts containing only trees amplitudes.

Using KLT relations N = 4 sYM results carry over immediately to N = 8 supergravity

$$M_n^{\text{tree}} = \sum_{i,j} g_{ij} A_n^{(i)} A_n^{(j)}$$

gravity

gauge



N=8 supergravity cuts are sums of products of N=4 sYM cuts

$$M_n^{L-\text{loop}}\Big|_{\text{cut}} = \sum_{\mathcal{N}=8} M_{n_1}^{\text{tree}} M_{n_2}^{\text{tree}} = \sum_{\mathcal{N}=8} \left(\sum_{i,j} g_{ij} A_{n_1}^{(i)} A_{n_1}^{(j)}\right) \left(\sum_{k,l} g_{kl} A_{n_2}^{(k)} A_{n_2}^{(l)}\right)$$
$$= \sum_{i,j,k,l} g_{ij} g_{kl} \left(\sum_{\mathcal{N}=4} A_{n_1}^{(i)} A_{n_2}^{(k)}\right) \left(\sum_{\mathcal{N}=4} A_{n_1}^{(j)} A_{n_2}^{(l)}\right)$$

- **On the cut N=8 supergravity is two copies of N=4 super-YM**
- **To understand N = 8 supergravity good to first understand N = 4 super-Yang-Mills theory.**

N = 4 Super-Yang-Mills Warmup: Results

Bern, Carrasco, LD, Johansson, Roiban, to appear

Consider UV divergences in critical dimension: $D_c = 6/L + 4$

two loops

$$\begin{aligned}\mathcal{A}_4^{(2)}(1,2,3,4)|_{\text{pole}} \;=\; & \frac{g^6\,\pi\,\mathcal{K}}{20\,(4\pi)^7\,\epsilon}\Big[(N_c^2+20)\big(s_{12}\,(\mathrm{Tr}_{1324}+\mathrm{Tr}_{1423}) \\ & \qquad +s_{23}\,(\mathrm{Tr}_{1243}+\mathrm{Tr}_{1342})+s_{13}\,(\mathrm{Tr}_{1234}+\mathrm{Tr}_{1432})\big) \\ & \qquad -20N_c\,(s_{12}\,\mathrm{Tr}_{12}\,\mathrm{Tr}_{34}+s_{23}\,\mathrm{Tr}_{14}\,\mathrm{Tr}_{23}+s_{13}\,\mathrm{Tr}_{13}\,\mathrm{Tr}_{24})\Big]\end{aligned}$$

$\partial^2\,\mathrm{Tr}F^4$ $\quad$ $\partial^2[\,\mathrm{Tr}F^2]^2$ $\quad$ **counterterms**

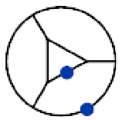
three- loops

$$\begin{aligned}\mathcal{A}_4^{(3)}(1,2,3,4)|_{\text{pole}} \;=\; & -\frac{g^8\,\mathcal{K}}{3\,(4\pi)^9\,\epsilon}(N_c^3+36\,\zeta(3)\,N_c)\,\Big[s_{12}\,(\mathrm{Tr}_{1324}+\mathrm{Tr}_{1423}) \\ & \qquad +s_{23}\,(\mathrm{Tr}_{1243}+\mathrm{Tr}_{1342})+s_{13}\,(\mathrm{Tr}_{1234}+\mathrm{Tr}_{1432})\Big]\end{aligned}$$

$\partial^2\,\mathrm{Tr}F^4$ $\quad$ $\cancel{\partial^2[\,\mathrm{Tr}F^2]^2}$ $\quad$ **counterterms**

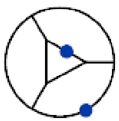
Four-Loop N = 4 super-YM Structure

Describe in terms of vacuum integrals



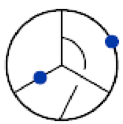
V_1

$$V_1 = \frac{1}{(4\pi)^{11}\,\epsilon}\left[\frac{512}{5}\,\Gamma^4(\tfrac{3}{4}) - \frac{2048}{105}\,\Gamma^3(\tfrac{3}{4})\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{4})\right] + \mathcal{O}(1)$$



V_2

$$V_2 = \frac{1}{(4\pi)^{11}\,\epsilon}\left[-\frac{4352}{105}\,\Gamma^4(\tfrac{3}{4}) + \frac{832}{105}\,\Gamma^3(\tfrac{3}{4})\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{4})\right] + \mathcal{O}(1)$$



V_8

$$V_8 = \frac{1}{(4\pi)^{11}}\frac{4}{21}\frac{1}{\Gamma(\tfrac{3}{4})}\frac{V_8^{\rm fin}}{\epsilon} \qquad V_8^{\rm fin} = 1.428452926283(3)$$

$$\begin{aligned}\mathcal{A}_4^{(4)}(1,2,3,4)|_{\rm pole} \;=\; & -6\,g^{10}\,\mathcal{K}\,N_c^2\left[N_c^2\,V_1 + 12\,(V_1 + 2\,V_2 + V_8)\right] \\ & \times\left[s_{12}\,(\mathrm{Tr}_{1324} + \mathrm{Tr}_{1423}) + s_{23}\,(\mathrm{Tr}_{1243} + \mathrm{Tr}_{1342}) \right.\\ & \left. + s_{13}\,(\mathrm{Tr}_{1234} + \mathrm{Tr}_{1432})\right]\end{aligned}$$

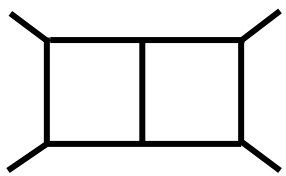
No double trace and no $N_c^{\,0}$ terms.

Double trace structure

See talk from Stelle for earlier alternative explanation. Also Berkovits, Green, Russo and Vanhove have an explanation.

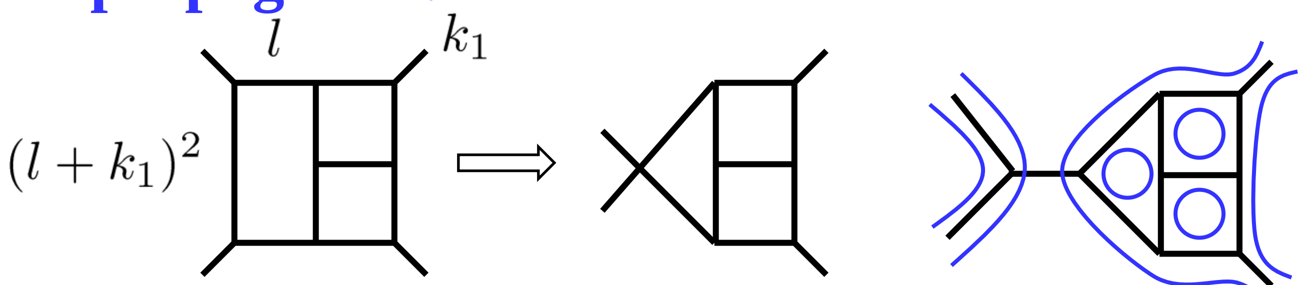
It is pretty simple to explain lack of double traces from amplitudes.

One and two loops: no contact 4 point contributions



Expand color factors: single and double color traces locked together trivially.

Three loops: leading behavior from diagrams with fewer loop propagators. Contact terms



***U*(1) decoupling kills double trace**

Extra symmetry wipes out double trace terms from the contact terms after permutation sum

First potential double counterterm has 2 extra derivatives
$D_c = 8/L + 4$ ($L > 2$) . A bit better than single trace.

Agreement for *N* = 4 super-Yang-Mills

Two other groups confirmed our results for sYM

- **Field theory algebraic nonrenormalization approach**
  See Kelly's talk **Bossard, Howe and Stelle (2009)**
- **String non-renormalization theorems analyzed in field theory limit.** **Berkovits, Green, Russo and Vanhove (2009)**

  - **Basic understanding unchanged since 1998**
  - **Recent slight improvement for double-trace terms.**

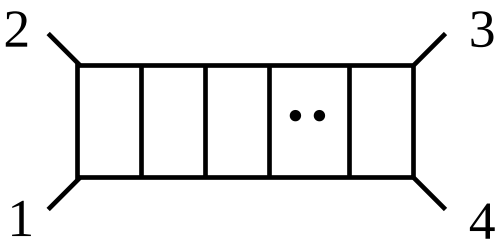
$$\boxed{D < \frac{6}{L} + 4} \qquad (L > 1)$$

UV finite in *D* = 4

***D*: dimension**
***L*: loop order**

It's likely that our understanding of UV properties of *N* = 4 sYM is basically settled given agreement.

L-Loops $N = 4$ Super-Yang-Mills Warmup

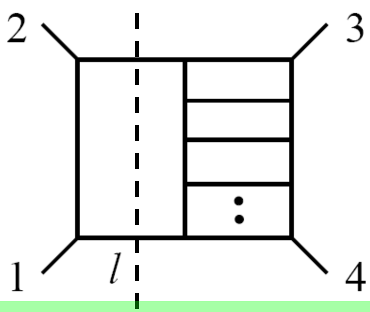


$$[(k_1 + k_2)^2]^{(L-2)}$$

numerator factor

ZB, Dixon, Dunbar, Perelstein, Rozowsky (1998)

From 2 particle cut:



$$[(l + k_4)^2]^{(L-2)}$$

numerator factor

Power counting this gives UV finiteness for :

$$D < \frac{6}{L} + 4$$

bound saturated for $L \leq 4$

Power count of UV behavior follows from supersymmetry alone.

- **Confirmed by explicit calculation through $L = 5$.**
- **Confirmed by Howe and Stelle using $N = 3$ harmonic superspace.**
- **Through $L = 6$ agrees with Berkovits, Green and Vanhove, who use low-energy limit of open string in Berkovits' pure spinor formalism.**
- **Though $L = 4$, *all* cancellations exposed by unitarity method!**

Novel N=8 Supergravity UV Cancellations

Consider instead *N* = 8 supergravity.

Will present a case that correct UV finiteness condition is:

$$\boxed{D < \frac{6}{L} + 4} \qquad (L > 1)$$

UV finite in *D* = 4
Same as *N* = 4 sYM!

D **: dimension**
L **: loop order**

Three pillars to our case:

- **Demonstration of *all*-loop order UV cancellations from no-triangle property.** ZB, Dixon, Roiban
- **Explicit 3,4 loop calculations.** ZB, Carrasco, Dixon, Johansson, Kosower, Roiban
- **Identification of tree-level cancellations responsible for improved UV behavior.** ZB, Carrasco, Ita, Johansson, Forde

Key claim: The most important cancellations are generic to gravity theories. Supersymmetry helps make the theory finite, but is *not* the key ingredient for finiteness.

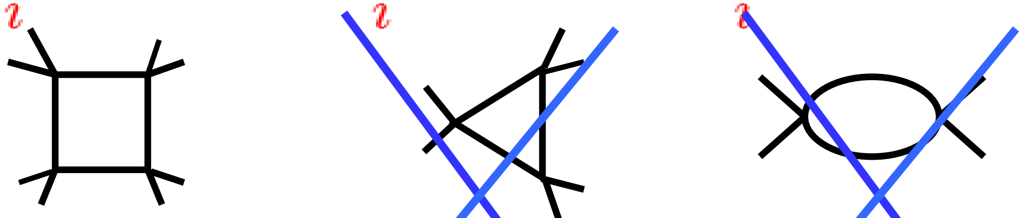
N = 8 Supergravity No-Triangle Property

ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr and Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Proofs by Bjerrum-Bohr and Vanhove; Arkani-Hamed, Cachazo and Kaplan.

One-loop D = 4 theorem: Any one loop amplitude is a linear combination of scalar box, triangle and bubble integrals with rational coefficients:

Brown, Feynman; Passarino and Veltman, etc

$$A_n^{\text{1-loop}} = \sum_i d_i I_4^{(i)} + \sum_i c_i I_3^{(i)} + \sum_i b_i I_2^{(i)}$$



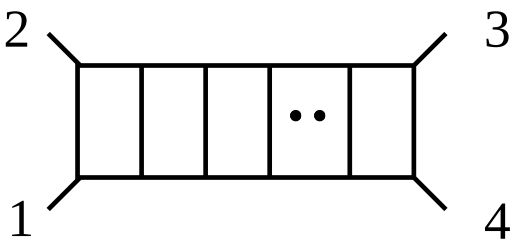
$$\int \frac{d^4p}{(p^2)^4} \qquad \cancel{\int \frac{d^4p}{(p^2)^3}} \qquad \cancel{\int \frac{d^4p}{(p^2)^2}}$$

- **In N = 4 Yang-Mills *only box* integrals appear. No triangle integrals and no bubble integrals.**
- **The "no-triangle property" is the statement that same holds in N = 8 supergravity. Non-trivial constraint on analytic form of amplitudes.**
- **Unordered nature of gravity is important for this property**

Bjerrum-Bohr and Vanhove

$\mathcal{N} = 8$ L-Loop UV Cancellations

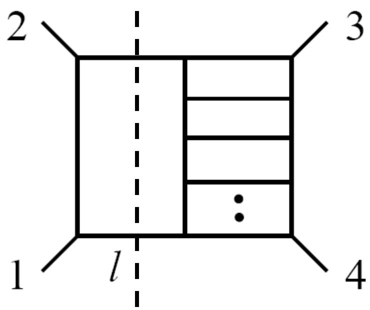
ZB, Dixon, Roiban



$$[(k_1 + k_2)^2]^{2(L-2)}$$

numerator factor

From 2 particle cut:

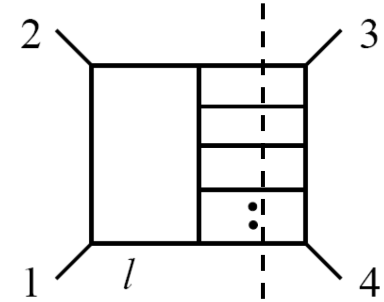


1 in $\mathcal{N} = 4$ YM

$$[(l + k_4)^2]^{2(L-2)}$$

numerator factor

L-particle cut



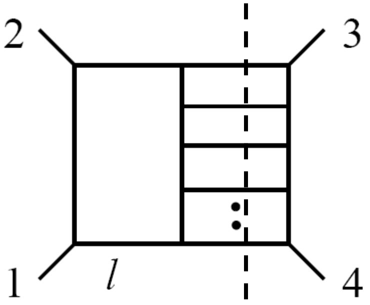
- **Numerator violates one-loop "no-triangle" property.**
- **Too many powers of loop momentum in one-loop subamplitude.**
- **After cancellations behavior is same as in $\mathcal{N} = 4$ Yang-Mills!**

- **UV cancellation exist to *all* loop orders! (not a proof of finiteness)**
- **These all-loop cancellations *not* explained by supersymmetry alone or by Berkovits' string theory non-renormalization theorem.**
- **Existence of these cancellations drive our calculations!**

Origin of Cancellations?

There does not appear to be a supersymmetry explanation for observed all-loop cancellations.

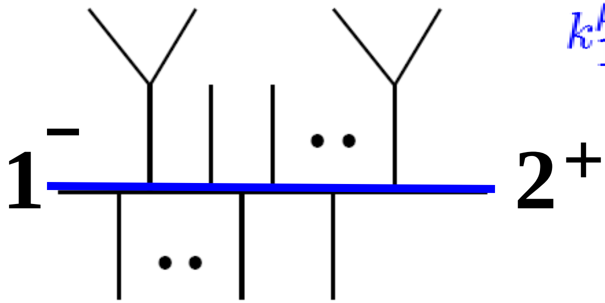
If it is *not* supersymmetrywhat might it be?



Origin of Cancellations?

First consider tree level

ZB, Carrasco, Forde, Ita, Johansson



$$k_1^\mu \to k_1^\mu + \frac{z}{2}\langle k_1^-|\gamma^\mu|k_2^-\rangle, \qquad k_2^\mu \to k_2^\mu - \frac{z}{2}\langle k_1^-|\gamma^\mu|k_2^-\rangle$$

***m* propagators and *m*+1 vertices between legs 1 and 2**

Yang-Mills scaling: $$z^{m+1} \times \frac{1}{z^m} \times \frac{1}{z^2} \sim \frac{1}{z}$$ **well behaved**

vertices **propagators** **polarizations**

$$z \to \infty$$

gravity scaling: $$z^{2(m+1)} \times \frac{1}{z^m} \times \frac{1}{z^4} \sim z^{m-2}$$ **poorly behaved**

Summing over all Feynman diagrams, correct gravity scaling is:

$$M_n^{\text{tree}}(z) \sim \frac{1}{z^2}$$

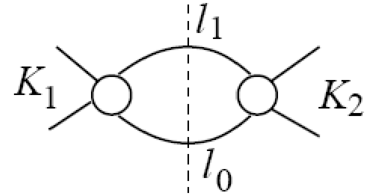
Remarkable tree-level cancellations.
Better than gauge theory!

z^{n-5} cancels to $\frac{1}{z^2}$

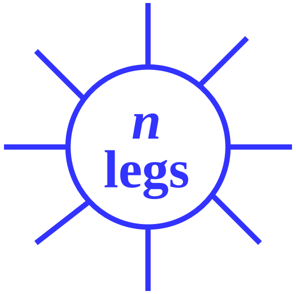
Bedford, Brandhuber, Spence, Travaglini;
Cachazo and Svrcek;
Benincasa, Boucher-Veronneau, Cachazo
Arkani-Hamed, Kaplan; Hall

Loop Cancellations in Pure Gravity

Powerful new one-loop integration method due to Forde makes it much easier to track the cancellations. Allows us to link one-loop cancellations to tree-level cancellations.



Observation: Most of the one-loop cancellations observed in *N* = 8 supergravity leading to "no-triangle property" are already present in non-susy gravity.



$$(l^\mu)^{2n} \rightarrow (l^\mu)^{n+4} \times (l^\mu)^{-8}$$

maximum powers of loop momenta

Cancellation generic to Einstein gravity

Cancellation from *N* = 8 susy

ZB, Carrasco, Forde, Ita, Johansson

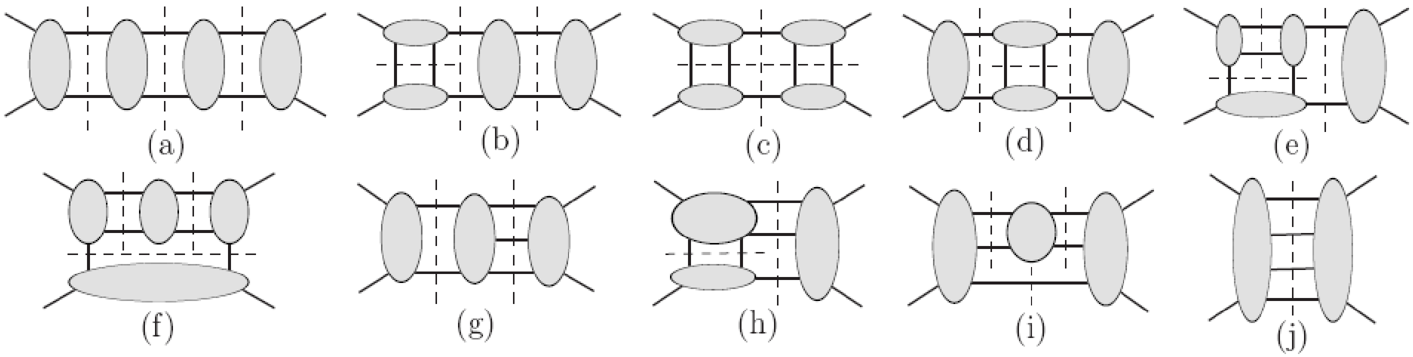
Proposal: This continues to higher loops, so that most of the observed *N* = 8 multi-loop cancellations are *not* due to susy but in fact are generic to gravity theories!

All-loop finiteness of *N* = 8 supergravity would follow from a combination of susy cancellations on top of novel but generic cancellations present even in pure Einstein gravity.

Full Three-Loop Calculation

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban

Need following cuts:



For cut (g) have:

reduces everything to product of tree amplitudes

$$\sum_{N=8\ \text{states}} M_4^{\text{tree}}(1,2,l_3,l_1) \times M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) \times M_5^{\text{tree}}(3,4,-q_1,-q_2,-q_3)$$

Use Kawai-Lewellen-Tye tree relations

$$M_4^{\text{tree}}(1,2,l_3,l_1) = -is_{12}A_4^{\text{tree}}(1,2,l_3,l_1)A_4^{\text{tree}}(2,1,l_3,l_1)$$

$$M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) = i\, s_{l_1q_1} s_{l_3q_3} A_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\, A_5^{\text{tree}}(-l_1,q_1,q_3,-l_3,q_2) + \{l_1 \leftrightarrow l_3\}\,,$$

supergravity

super-Yang-Mills

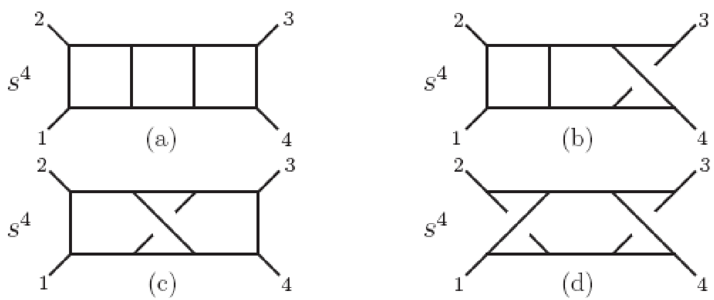
N = 8 supergravity cuts are sums of products of N = 4 super-Yang-Mills cuts

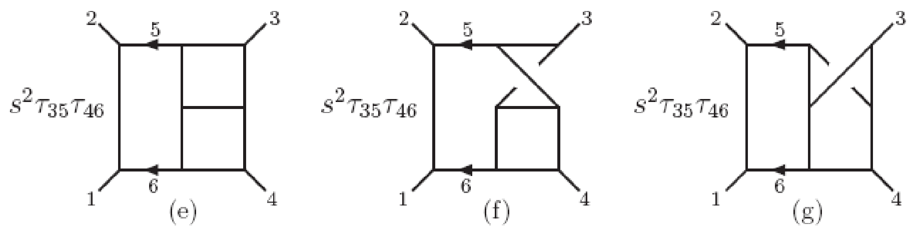
Complete Three-Loop $N = 8$ Supergravity Result

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban; hep-th/0702112
ZB, Carrasco, Dixon, Johansson, Roiban arXiv:0808.4112 [hep-th]

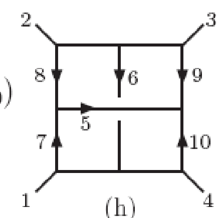
$$M_4^{(3)} = \left(\frac{\kappa}{2}\right)^8 stu M_4^{\text{tree}} \sum_{S_3} \left[I^{(a)} + I^{(b)} + \tfrac{1}{2} I^{(c)} + \tfrac{1}{4} I^{(d)} + 2 I^{(e)} + 2 I^{(f)} + 4 I^{(g)} + \tfrac{1}{2} I^{(h)} + 2 I^{(i)} \right]$$

$$\tau_{ij} = 2 k_i \cdot k_j$$

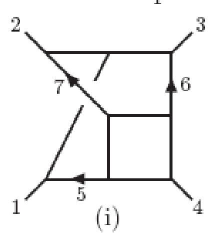




$$\begin{aligned}&(s(\tau_{26}+\tau_{36})+t(\tau_{15}+\tau_{25})+st)^2\\&+(s^2(\tau_{26}+\tau_{36})-t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})\\&+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10})+t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})\\&+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})\end{aligned}$$



$$\begin{aligned}&(s\tau_{45}-t\tau_{46})^2-\tau_{27}(s^2\tau_{45}+t^2\tau_{46})\\&-\tau_{15}(s^2\tau_{47}+u^2\tau_{46})-\tau_{36}(t^2\tau_{47}+u^2\tau_{45})\\&+l_5^2 s^2 t+l_6^2 s t^2-\tfrac{1}{3} l_7^2 stu\end{aligned}$$



Three loops is not only UV finite it is "superfinite"— cancellations beyond those needed for finiteness in $D = 4$.

Finite for $D < 6$

No term more divergent than the total amplitude. *All* cancellations exposed!

$$\text{UVpole}_{D=6-2\epsilon} = \frac{1}{\epsilon}\frac{5\zeta_3}{(4\pi)^9}\left(\frac{\kappa}{2}\right)^8 (stu)^2 M_4^{\text{tree}}$$

Identical manifest power count as $N = 4$ super-Yang-Mills

Four Loop $N = 8$ Supergravity

ZB, Carrasco, Dixon, Johansson, Roiban

- **No-triangle unitarity analysis predicts no divergence $D < 5.5$**

- **Prior algebric approach predicts divergence in $D = 5$.**

Bossard, Howe and Stelle

- **Berkovits string non-renormalization theorem also suggests $D < 5.5$. But we want to know the answer in field theory.**

Berkovits, Green, Russo and Vanhove

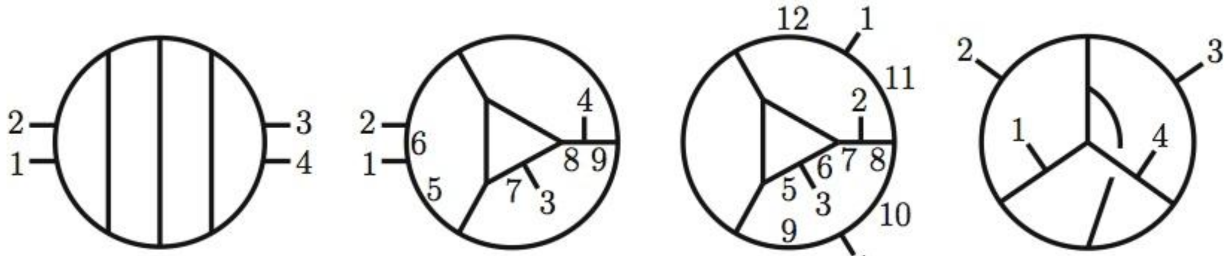
D^6R^4 would be expected counterterm in $D = 5$ if it diverges.

We have the tools to determine this decisively.

Four-Loop Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban

Get 50 distinct diagrams or integrals (ones with two- or three-point subdiagrams not needed).



I_1 I_{25} I_{32} I_{50}

Journal submission has mathematica files with all 50 diagrams

$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

Integral

leg perms

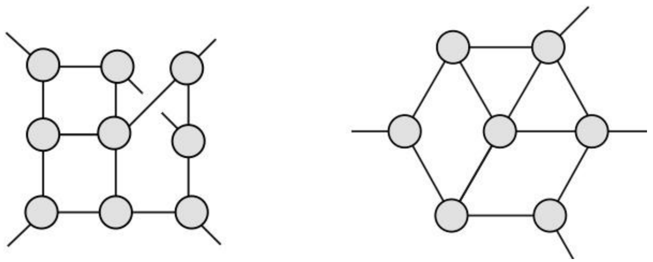
symmetry factor

Four-Loop Construction

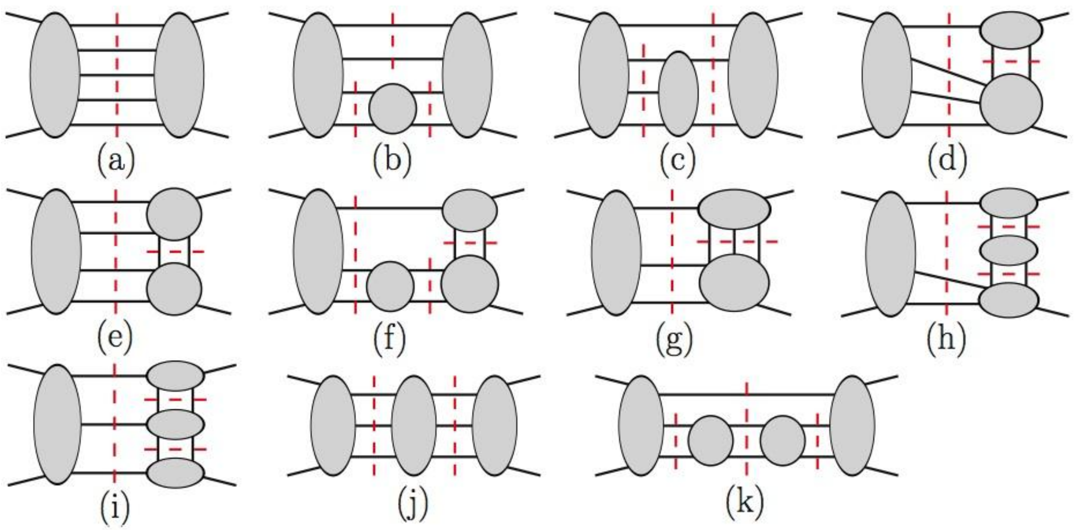
numerator

$$I_i = \int d^D l_1 d^D l_2 d^D l_3 d^D l_4 \, \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 l_6^2 l_7^2 l_8^2 l_9^2 l_{10}^2 l_{11}^2 l_{12}^2 l_{13}^2}$$

Determine numerators from 2906 maximal and near maximal cuts



Completeness of expression confirmed using 26 generalized cuts sufficient for obtaining the complete expression



11 most complicated cuts shown

UV Finiteness at Four Loops

$$I_i = \int d^D l_1 d^D l_2 d^D l_3 d^D l_4 \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 l_6^2 l_7^2 l_8^2 l_9^2 l_{10}^2 l_{11}^2 l_{12}^2 l_{13}^2}$$

$$N_i \sim O(k^4 l^8)$$

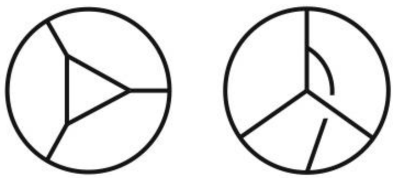
k_i**: external momenta**

l_i **: loop momenta**

The N_i are rather complicated objects, but it is straightforward to analyze UV divergences.

Manfestly finite for $D = 4$, but no surprise here.

Leading terms can be represented by two vacuum diagrams which cancel in the sum over all contributions.



coefficients vanish

$O(k^4 l^8)$

- **If no further cancellation corresponds to $D = 5$ divergence.**

UV Finiteness in D = 5 at Four Loops

ZB, Carrasco, Dixon, Johansson, Roiban

$N \sim O(k^6 l^6)$ **corresponds to D = 5 divergence.**

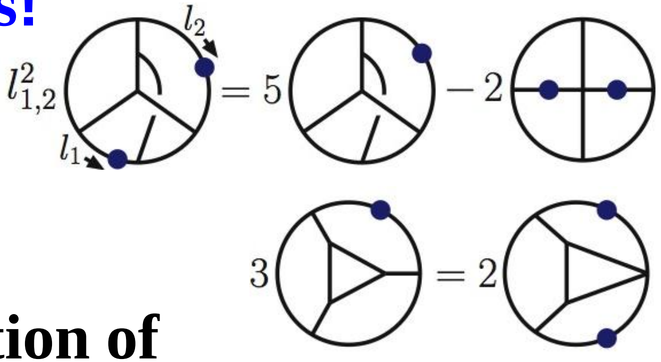
Expand numerator and propagators in small k

$$\frac{1}{(l_j + K_n)^2}$$

$$N^{(6)} + N^{(7)} \frac{K_i \cdot l_j}{l_j^2} + N^{(8)} \left(\frac{K_i^2}{l_j^2} + \frac{K_i \cdot l_j \; K_m \cdot l_n}{l_j^2 l_n^2} \right)$$

Marcus & Sagnotti
UV extraction method

Cancels after using D = 5 integral identities!



UV finite for D = 4 and 5

actually finite for D < 5.5

1. **Shows potential supersymmetry explanation of three loop result by Bossard, Howe, Stelle does *not* work!**
2. **The cancellations are stronger at 4 loops than at 3 loops, which is in turn stronger than at 2 loops. Rather surprising from traditional susy viewpoint.**

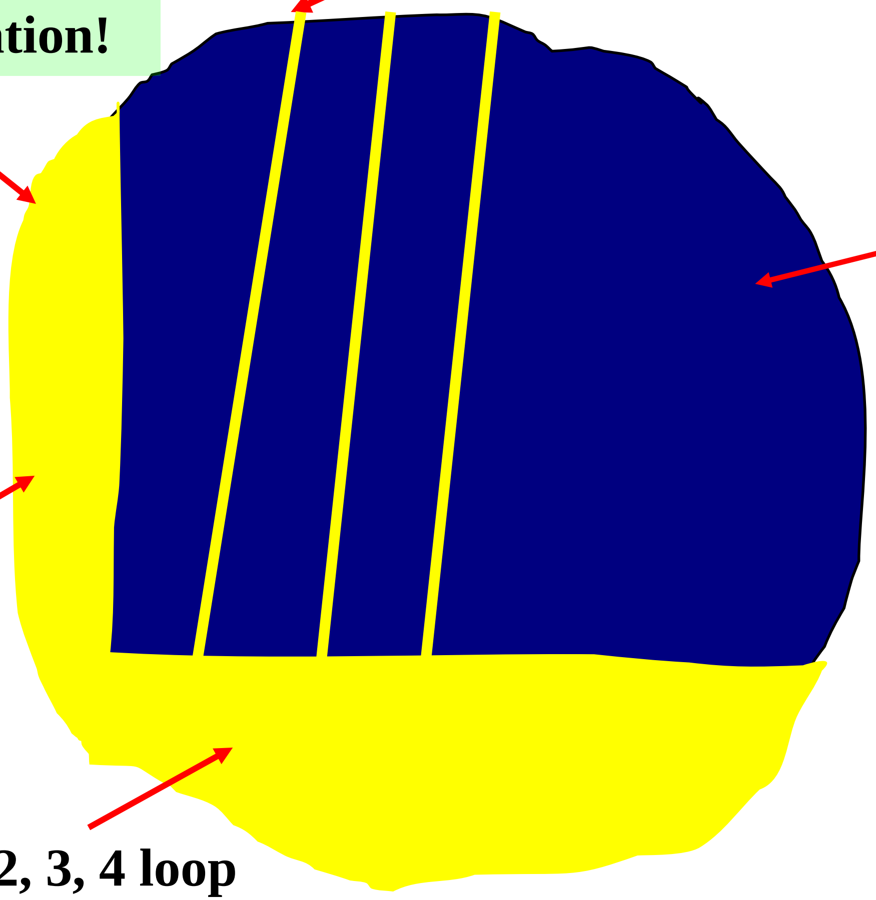
see Kelly's talk for update

Schematic Illustration of Status

- Same power count as *N*=4 super-Yang-Mills
- UV behavior unknown

All-loop UV finiteness.
No susy explanation!

from feeding 2, 3 and 4 loop calculations into iterated cuts.



finiteness unproven

loops

No triangle property

explicit 2, 3, 4 loop computations

terms

Non-perturbative issues

This started a debate on the possible nonperturbative consistency of a finite theory of N = 8 supergravity.

- **Ooguri, Green and Schwarz argued that N = 8 supergravity can’t be nonperturbatively smoothly connected to string theory.**
- **Banks and Strominger argue for generic reasons massive blackholes can become massless leading to inconsistencies, due to the appearance of singularities.**
- **Bianchi, Ferrara, Kallosh argue back that if you actually study the spectrum, inconsistencies of Banks and Strominger not present.**

It will be interesting to see how this turns out

Future

- Can we construct a finiteness proof? Make use of iterative structure of the cuts.
- Other theories. Our expectation is theories with N > 4 susy will be UV finite if N = 8 is finite.
- Detailed understanding of the origin of the cancellations.
- Non-perturbative issues?

I don't know if we will be able to find a completely satisfactory description of Nature via supergravity, but what is clear is that the possibility of doing so is back from a 25 year coma.

Summary

- **Modern unitarity method gives us means to calculate at high loop order. We can explicitly check claims. We can peer to *all* loop orders.**

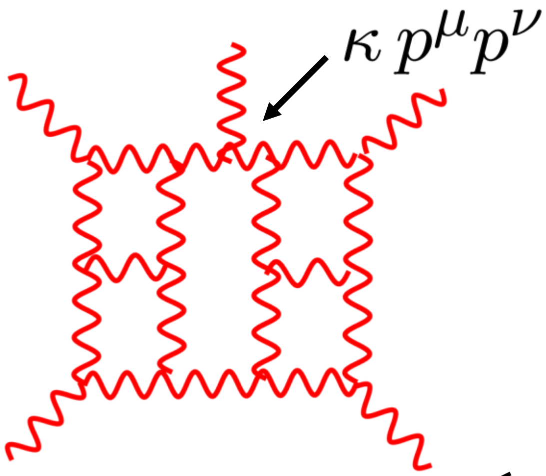
- **Gravity ~ (gauge theory) x (gauge theory) at tree level.**

- **N = 8 supergravity has ultraviolet cancellations with no known supersymmetry explanation.**
  - **No-triangle property implies cancellations strong enough for finiteness to *all* loop orders, in a limited class of terms.**
  - **At four points three and four loops, *established* that cancellations are complete and N = 8 supergravity same UV power counting as N = 4 Yang-Mills theory.**
  - **Key cancellations appear to be generic in gravity.**

N = 8 supergravity may well be the first example of a unitary point-like perturbatively UV finite theory of gravity. Demonstrating this remains a challenge.

Extra Transparancies

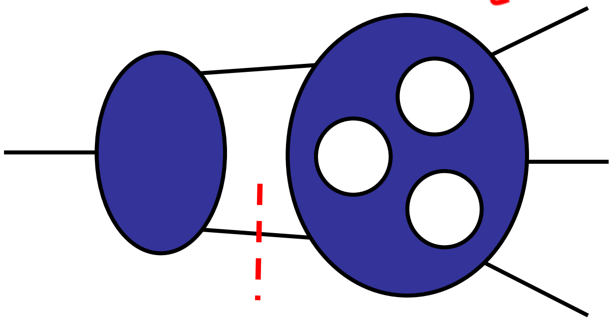
Higher-Point Divergences?



Add an extra leg:

1. **extra $\kappa\, p^\mu p^\nu$ in vertex**
2. **extra $1/p^2$ from propagator**

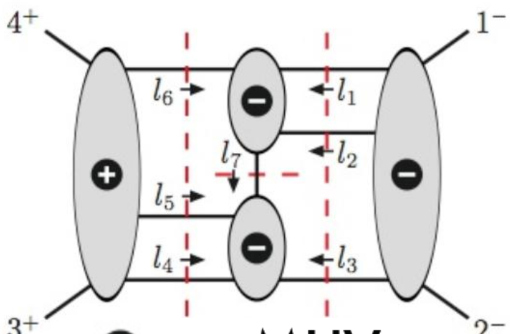
Adding legs generically does not worsen power count.



Cutting propagators exposes lower-loop higher-point amplitudes.

- **Higher-point divergences should be visible in high-loop four-point amplitudes.**
- **A proof of UV finiteness would need to systematically rule out higher-point divergences.**

Susy Sneakiness



⊕ = MHV

⊖ = $\overline{\text{MHV}}$

Exploit similarity of QCD (pure glue) and *N*=4 sYM numerators

QCD: $A^4 + B^4 + C^4 + D^4 + E^4 + F^4 + G^4 + H^4$

$N = 4$ sYM: $(A + B + C + D + E + F + G + H)^4$

$$A = \langle l_4\, l_5\rangle\, [l_4\, l_5]\, [l_2\, l_7]\, [l_1\, l_3]\;, \quad B = \langle l_4\, l_5\rangle\, [l_4\, l_5]\, [l_7\, l_1]\, [l_2\, l_3]\;, \quad C = \langle l_4\, l_6\rangle\, [l_4\, l_7]\, [l_2\, l_6]\, [l_1\, l_3]\;,$$

$$D = \langle l_4\, l_6\rangle\, [l_4\, l_7]\, [l_6\, l_1]\, [l_2\, l_3]\;, \quad E = \langle l_5\, l_6\rangle\, [l_5\, l_7]\, [l_2\, l_6]\, [l_1\, l_3]\;, \quad F = \langle l_5\, l_6\rangle\, [l_5\, l_7]\, [l_6\, l_1]\, [l_2\, l_3]\;,$$

$$G = \langle l_4\, l_6\rangle\, [l_2\, l_1]\, [l_3\, l_4]\, [l_6\, l_7]\;, \quad H = \langle l_5\, l_6\rangle\, [l_2\, l_1]\, [l_3\, l_5]\, [l_6\, l_7]\;.$$

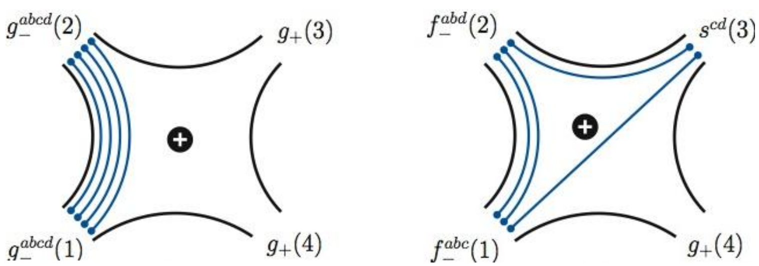
$$\Big[A + B + C + D + E + F + G + H\Big]^4 = \Big[s\, [l_1\, l_2]\, [l_7\, l_3]\Big]^4$$

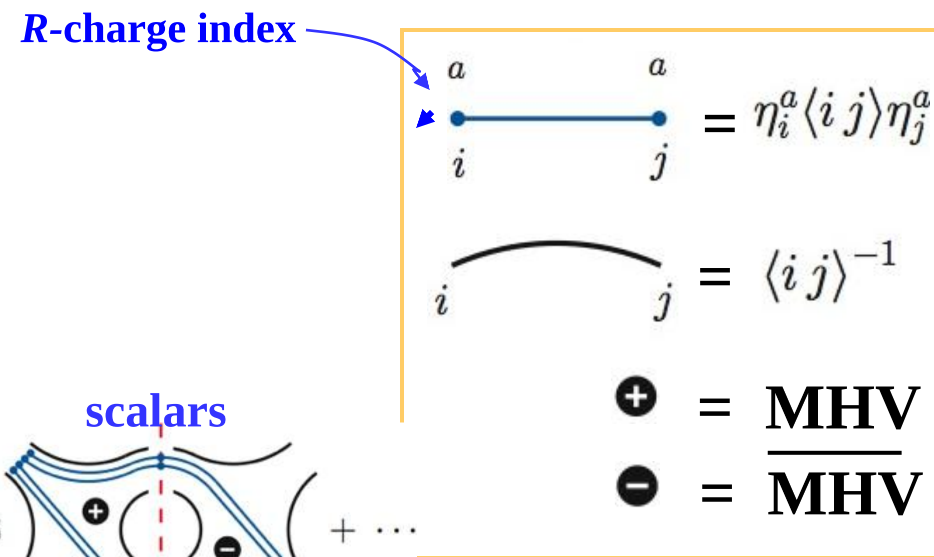
8^4 = 4096 individual contributions resummed

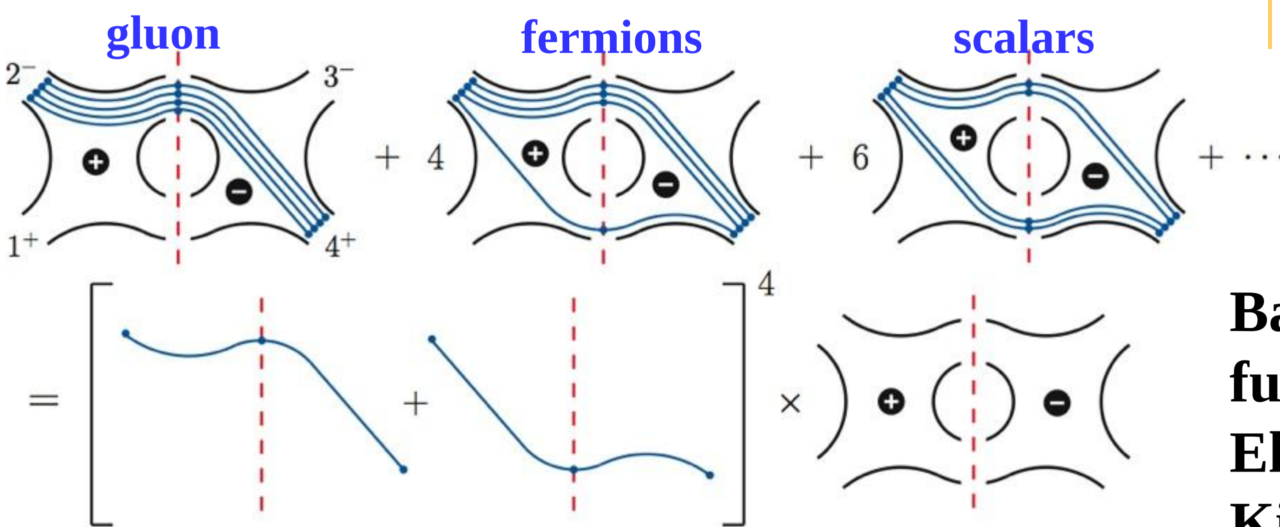
Cut with non-MHV amplitudes follow easily from MHV vertex expansion. Again everything deduced from pure gluons.

R-Index Diagrams: Sneakiness Justified

ZB, Carrasco, Johansson, Ita, Roiban







Based on generating function of Bianchi, Elvang , Freedman, Kiermaier

- **Sum over paths gives numerators (*A* + *B* + *C* + …)^4**
- **Tracks individual particles.**

Comments on Consequences of Finiteness

- Suppose N = 8 SUGRA is finite to all loop orders. Would this prove that it is a **nonperturbatively** consistent theory of quantum gravity? **Of course not!**
- At least two reasons to think it needs a nonperturbative completion:
  - — Likely $L!$ or worse growth of the order L coefficients,

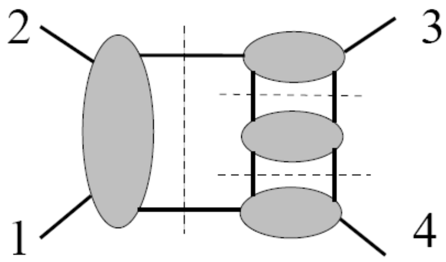
    $$\sim L! \ (s/M_{\mathrm{Pl}}^2)^L$$
  - — Different $E_{7(7)}$ behavior of the perturbative series (invariant!), compared with the $E_{7(7)}$ behavior of the mass spectrum of black holes (non-invariant!)
- Note QED is renormalizable, but its perturbation series has **zero radius of convergence in α**: $\sim L! \ \alpha^L$. But it has many point-like nonperturbative UV completions —asymptotically free GUTS.

Power Counting To All Loop Orders

Z.B., Dixon, Dunbar, Perelstein and Rozowsky

From '98 paper:

- **Assumed rung-rule contributions give the generic UV behavior.**
- **Assumed no cancellations with other uncalculated terms.**



- **No evidence was found that more than 12 powers of numerator momenta come out of the integrals as external momenta.**
- **No improvement seen on cancellations found at two loops.**

Elementary power counting for 12 external momenta and the rest loop momenta gives finiteness condition:

$$\boxed{D < \frac{10}{L} + 2} \qquad (L > 1)$$

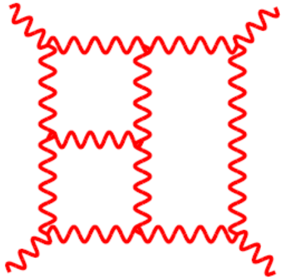
In *D* = 4 finite for *L* < 5.
***L* is number of loops.**

$D^4 R^4$ **counterterm expected in *D* = 4, for** $L = 5$

Feynman Diagrams for Gravity

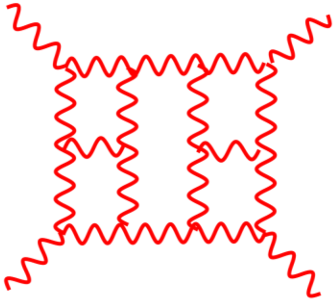
Suppose we want to put an end to the speculations by explicitly calculating to see what is true and what is false:

What would happen if we tried with Feynman diagrams?



If we attack this directly get $\sim 10^{20}$ **terms in diagram. There is a reason why this hasn't been evaluated using Feynman diagrams.**

In 1998 we suggested that five loops is where the divergence is:



This single diagram has $\sim 10^{30}$ **terms prior to evaluating any integrals.**
More terms than atoms in your brain!

Method of Maximal Cuts

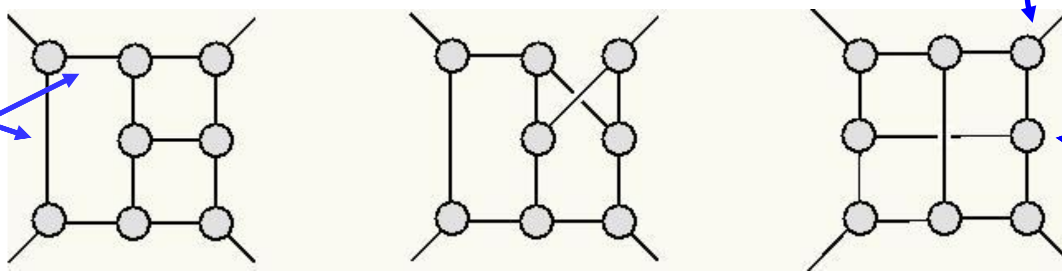
ZB, Carrasco, Johansson, Kosower

A refinement of unitarity method for constructing complete higher-loop amplitudes is "Method of Maximal Cuts". Systematic construction in any massless theory.

To construct the amplitude we use cuts with maximum number of on-shell propagators:

tree amplitudes

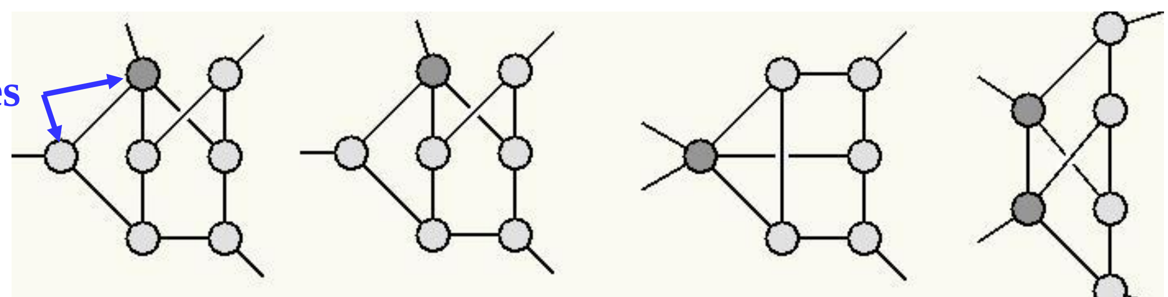
on-shell



Maximum number of propagator placed on-shell.

Then systematically release cut conditions to obtain contact terms:

tree amplitudes



Fewer propagators placed on-shell.

Related to subsequent leading singularity method which uses hidden singularities.

Cachazo and Skinner; Cachazo; Cachazo, Spradlin, Volovich; Spradlin, Volovich, Wen

More Recent Opinion

In a recent paper Bossard, Howe and Stelle had a careful look at the question of how much supersymmetry can tame UV divergences.

In particular, they [non-renormalization theorems and algebraic formalism] suggest that maximal supergravity is likely to diverge at four loops in $D = 5$ and at five loops in $D = 4$, unless other infinity suppression mechanisms not involving supersymmetry or gauge invariance are at work. Bossard, Howe, Stelle (2009)

D^6R^4 would be expected counterterm in $D = 5$.

We have the tools to decide this decisively.

Confirmation of Results

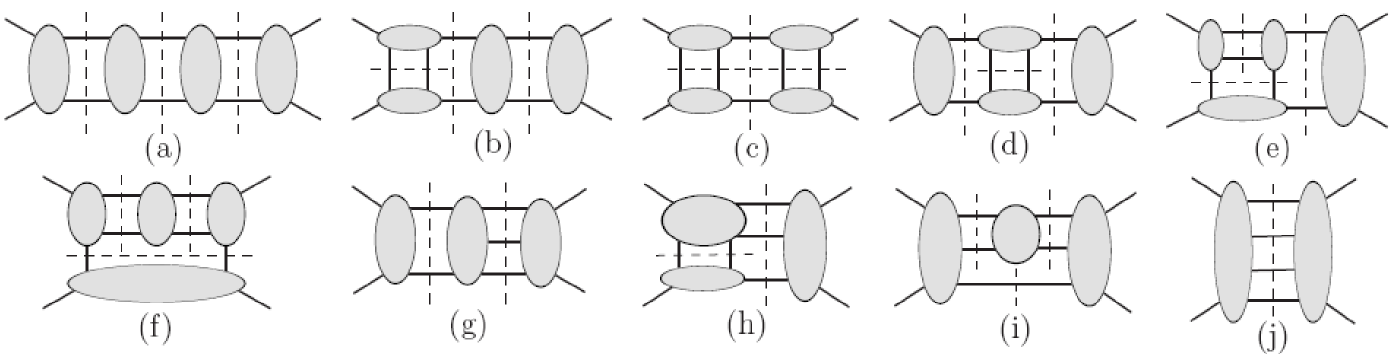
A technicality:

• ***D*= 4 kinematics used in maximal cuts – need *D* dimensional cuts. Pieces may otherwise get dropped.**

Once we have an ansatz from maximal cuts, we confirm using more standard generalized unitarity in *D* dimensions.

ZB, Dixon, Kosower

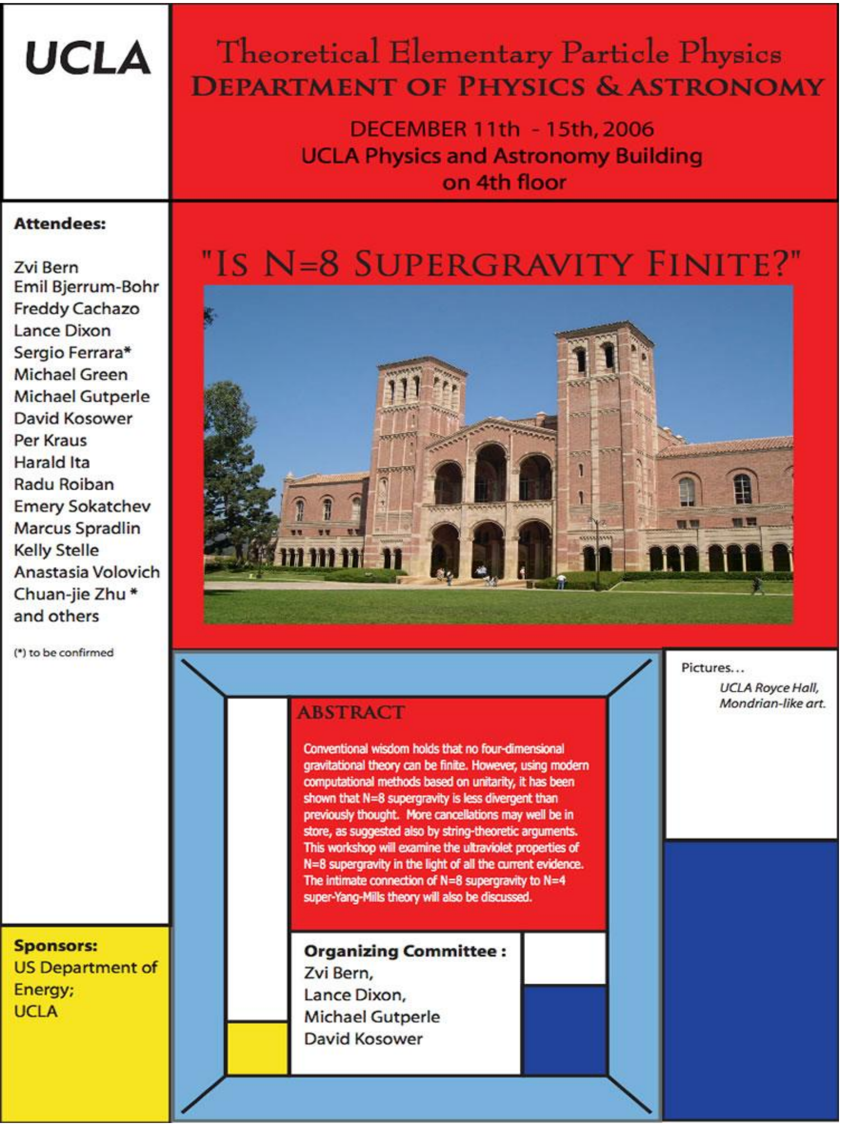
At three loops, following cuts guarantee nothing is lost:



$\mathcal{N} = 1$, $D = 10$ sYM equivalent to $\mathcal{N} = 4$, $D = 4$

Where it re-started

In mid-2006 we decided to organize a conference at UCLA entitled "Is $N = 8$ Supergravity Finite?"



Wonderful progress in past 3 years!

Conference organized without published evidence for UV finiteness!

Known no-triangle property together with unitarity method made it a safe bet! Novel UV cancellations had to exist to all loop orders.