Non-renormalization Theorems for Maximal Supersymmetric Theories

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G. Bossard, P.S. Howe & K.S.S., arXiv 0901.4661 & 0908.3883 [hep-th]

A bet with Zvi Bern on D=5, L=4 maximal supergravity divergences was lost and paid off in Rome on 26 June, 2009.

- 2001 Barolo,

  G.D. Vajra producer

  consultant:

  E. Kiritsis

Forward

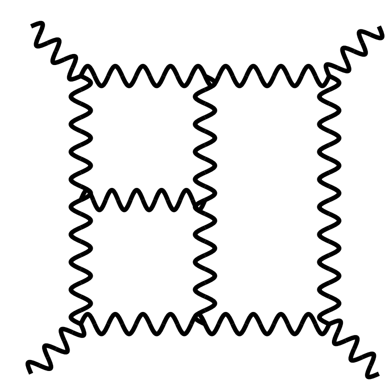
- From the non-renormalization theorem point of view, this story is about potentially divergent structures that need to be written as subsurface integrals of the full on-shell superspace of a given theory, aka "F terms."
- The whole question is which F terms are allowed by the Ward identities and which ones are ruled out by non-renormalization theorems.
- Of course, eventually, one will reach a loop order where D terms, *i.e.* full superspace integrals, are possible, and these do not appear to be ruled out by supersymmetry, although one can contemplate the rôles of other symmetries such as duality symmetries.

Ultraviolet Divergences in Gravity

- Simple power counting in gravity and supergravity theories leads to a naïve degree of divergence

$$\Delta = (D-2)L+2$$

in D spacetime dimensions. So, for D=4, L=3, one expects $\Delta = 8$. In dimensional regularization, only logarithmic divergences are seen ($\frac{1}{\epsilon}$ poles, $\epsilon = D-4$), so 8 powers of momentum would have to come out onto the external lines of such a diagram.



- Local supersymmetry implies that the pure curvature part of such a D=4, 3-loop divergent structure must be built from the square of the Bel-Robinson tensor

Deser, Kay & K.S.S

$$\int \sqrt{-g} T_{\mu\nu\rho\sigma} T^{\mu\nu\rho\sigma} \,, \qquad T_{\mu\nu\rho\sigma} = R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} R_{\rho\alpha\sigma\beta} + {}^{*}R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\, {}^{*}R_{\rho\alpha\sigma\beta}$$

- This is directly related to the α'^3 corrections in the superstring effective action, except that in the string context such contributions occur with finite coefficients. The question remains whether such string theory contributions develop poles in $(\alpha')^{-1}$ as one takes the zero-slope limit $\alpha' \to 0$ and how this bears on the ultraviolet properties of the corresponding field theory.

Berkovits; Green, Russo & Vanhove

- The consequences of supersymmetry for the ultraviolet structure are not restricted, however, simply to the requirement that counterterms be supersymmetric invariants.

- There exist more powerful "non-renormalization theorems," the most famous of which excludes infinite renormalization within D=4, N=1 supersymmetry of chiral invariants, given in N=1 superspace by integrals over half the superspace:

$$\int d^2\theta W(\phi(x,\theta,\bar{\theta}))\ , \quad \bar{D}\phi = 0$$

- The strength of a given supersymmetric non-renormalization theorem depends on the extent of linearly realizable, or "off-shell" supersymmetry. This is the extent of supersymmetry for which the algebra can close without use of the equations of motion.

- Knowing the extent of this off-shell supersymmetry is tricky, and may involve formulations (*e.g.* harmonic superspace) with infinite numbers of auxiliary fields.

  Galperin, Ivanov, Kalitsin, Ogievetsky & Sokatchev

- For maximal N=4 Super Yang-Mills and maximal N=8 supergravity, the linearly realizable supersymmetry has been known since the 1980's to be at least half the full supersymmetry of the theory.

- The full extent of a theory's supersymmetry, even though it may be non-linear, also restricts the infinities since the *leading* counterterms have to be invariant under the original unrenormalized supersymmetry transformations.
- Assuming that 1/2 supersymmetry is linearly realizable and requiring gauge and supersymmetry invariances, predictions were derived for the first divergent loop orders in maximal (N=4 ↔ 16 supercharge) SYM and (N=8 ↔ 32 sc.) SUGRA:

Howe, K.S.S & Townsend

Max. SYM first divergences, assuming half SUSY off-shell (8 supercharges)

Dimension D	10	8	7	6	5	4
Loop order L	1	1	2	3	4	∞
Gen. form	$\partial^2 F^4$	F^4	$\partial^2 F^4$	$\partial^2 F^4$	F^4	finite

Max. SUGRA first divergences, assuming half SUSY off-shell (16 supercharges)

Dimension D	11	10	8	7	6	5	4
Loop order L	2	2	1	2	3	2	3
Gen. form	$\partial^{12} R^4$	$\partial^{10} R^4$	R^4	$\partial^4 R^4$	$\partial^6 R^4$	R^4	R^4

Unitarity-based calculations

Bern, Carrasco, Dixon, Dunbar, Johansson, Kosower, Perelstein, Roiban, Rozowsky et al.

- Within the last decade, there have been significant advances in the computation of loop corrections in quantum field theory.
- These developments include the organization of amplitudes into a new kind of perturbation theory starting with maximal helicity violating amplitudes (MHV), then next-to-MHV (NMHV), *etc.*
- They also incorporate a specific use of dimensional regularization together with a clever use of unitarity cutting rules.

- Normally, one thinks of unitarity relations such as the optical theorem as giving information only about the imaginary parts of amplitudes. However, if one keeps all orders in an expansion in $\epsilon = D - 4$, then loop integrals like $\int d^{(4+\epsilon)}p$ require integrands to have an additional momentum dependence $f(s) \to f(s)s^{-\epsilon/2}$, where s is a momentum invariant. Then, since $s^{-\epsilon/2} = 1 - (\epsilon/2)\ln(s) + \ldots$ and $\ln(s) = \ln(|s|) + i\pi\Theta(s)$, one can learn about the real parts of an amplitude by retaining imaginary terms at order ϵ .

- This gives rise to a procedure for the *cut construction* of higher-loop diagrams.

- Key links between maximal supergravity and maximal SYM are the Kawai-Lewellen-Tye (KLT) relations between open- and closed-string amplitudes. These give rise to tree-level relations between field-theoretic max. SUGRA and max. SYM field-theory amplitudes, *e.g.*

$$M_4^{\rm tree}(1,2,3,4) = -is_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)$$

- Combining this with the unitarity-based calculations, in which all amplitudes are ultimately reduced to integrals over products of tree amplitudes, one has a way to obtain higher-loop supergravity amplitudes from SYM amplitudes.

- In this way, a different set of anticipated first loop orders for ultraviolet divergences arose using the unitarity-based approach, circa 1998-2000:

Max. SYM first divergences, early unitarity-based predictions

Dimension D	10	8	7	6	5	4
Loop order L	1	1	2	3	6	∞
Gen. form	$\partial^2 F^4$	F^4	$\partial^2 F^4$	$\partial^2 F^4$	$\partial^2 F^4$	finite

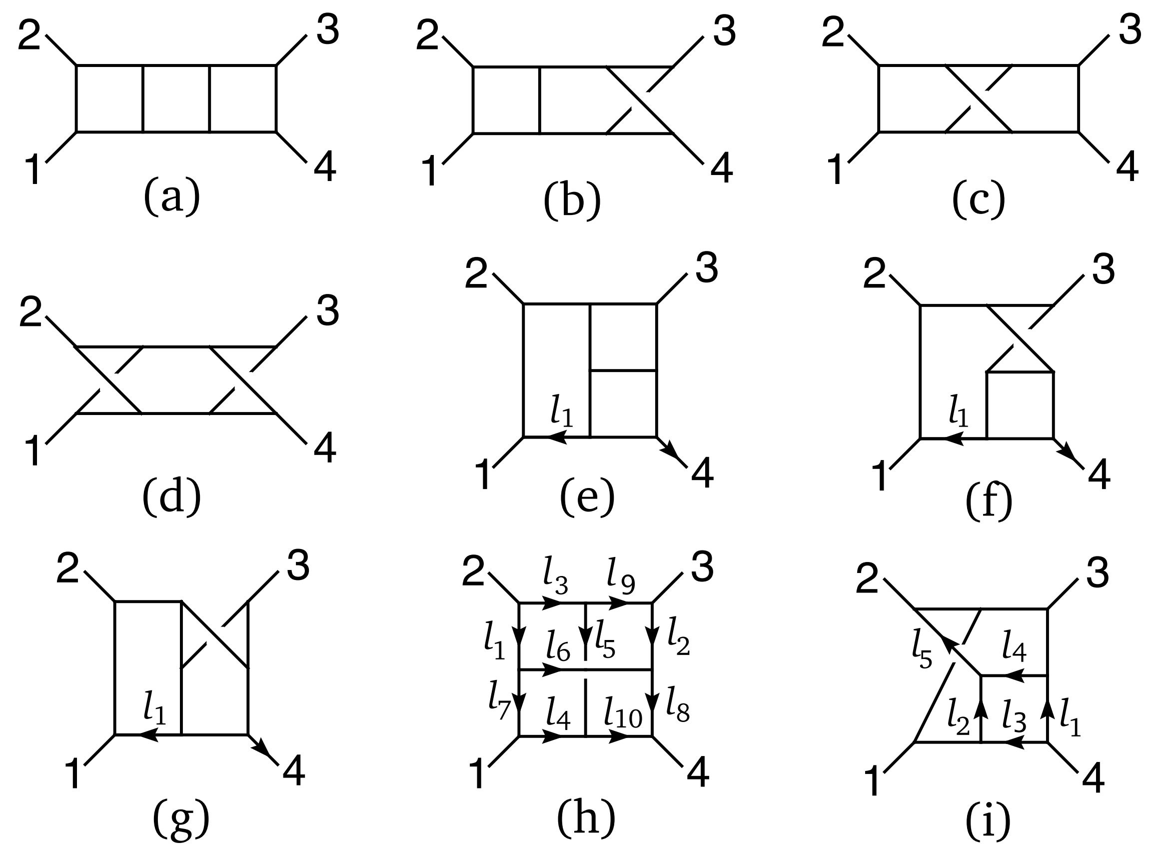
Max. SUGRA first divergences, early unitarity-based predictions

Dimension D	11	10	8	7	6	5	4
Loop order L	2	2	1	2	3	4	5
Gen. form	$\partial^{12} R^4$	$\partial^{10} R^4$	R^4	$\partial^4 R^4$	$\partial^6 R^4$	$\partial^6 R^4$	$\partial^4 R^4$

- These anticipations were based on iterated 2-particle cuts, however. Full calculations can reveal different behavior, however.

- An important development was the subsequent completion of the 3-loop calculation: Bern, Carrasco, Dixon, Johansson, Kosower & Roiban.

Normal Feynman diagram calculation of these would involve about 10^{20} terms

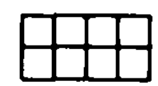
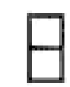


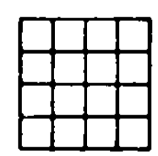
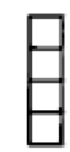
- Diagrams (a-g) can be evaluated using iterated two-particle cuts, but diagrams (h) & (i) cannot. The result is finite at L=3 in D=4, but a surprize was that the finite parts have an unexpected six powers of momentum that come out onto the external lines, giving a $\partial^6 R^4$ leading effective action correction.

Counterterm analysis

- The 3-loop N=8 supergravity calculation is a remarkable *tour de force*, but does it indicate that there are "miracles" that cannot be understood from non-renormalization theorems?
- All single-trace SYM divergences in the various dimensions D can be understood using non-renormalization theorems.
- Recently it has been realized that N=4 SYM can also be quantized with 9=8+1 off-shell supersymmetries, at the price of manifest Lorentz invariance. Baulieu, Berkovits, Bossard & Martin
- A similar formulation for maximal supergravity exists with 17=16+1 off-shell supersymmetries in D=2. Indications are that a related construction rules out the L=3, D=4 SG counterterm. Bossard, Howe & K.S.S

- The ***8+1*** max. SYM and the ***16+1*** max. SG formalisms allow one now to counter the eligibility of counterterms involving integration over half the corresponding full on-shell superspaces, *i.e.* 8 integrations for SYM and 16 for SG. These two "half BPS" counterterms have similar D=4 structures:

$$\Delta I_{SYM} = \int (d^4\theta d^4\bar{\theta})_{105} \mathrm{tr}(\phi^4)_{105}$$  ***105*** ϕ_{ij}  ***6*** of SU(4)

$$\Delta I_{SG} = \int (d^8\theta d^8\bar{\theta})_{232848} (W^4)_{232848}$$  ***232848*** W_{ijkl}  ***70*** of SU(8)

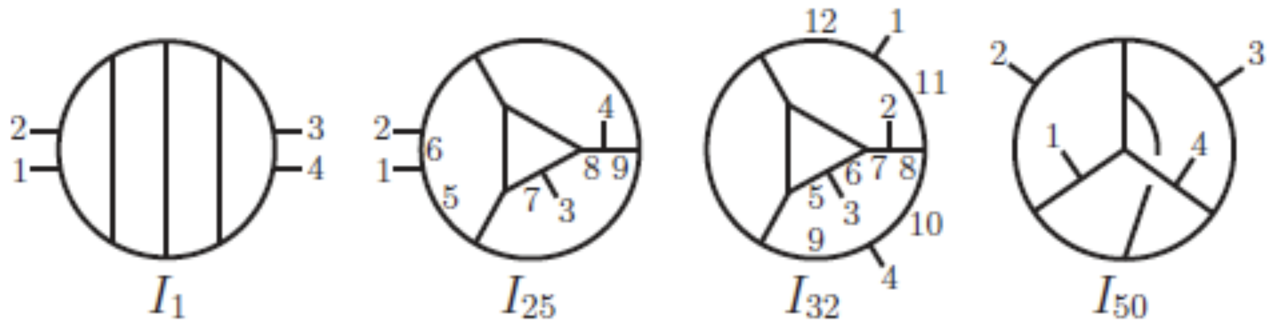
Kallosh

Howe, K.S.S. & Townsend

- Assuming that non-renormalization theorems work similarly to all other known cases, the "half SUSY +1" formalisms are just enough to rule out the 1/2 BPS F^4 SYM and R^4 SG counterterms.

Meanwhile, the 4-loop calculation has now been done (May 2009).

Bern, Carrasco, Dixon, Johansson & Roiban



+ 46 more topologies

- Result: $M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$ is ultraviolet finite in D=4 (as expected) *and* in D=5 (unexpected).
- One bottle of wine has been lost.

Super Yang-Mills analogue

Bern, Carrasco, Dixon, Johansson & Roiban

- A surprizing thing about the D=5, L=4 max. supergravity divergence cancellation is that its naïve degree of divergence $\Delta = 14 \leftrightarrow \partial^6 R^4$ is the *same* as for D=6, L=3, where a divergence *does* occur.
- There is a similar puzzle in max. SYM, contrasting D=7, L=2 with D=6, L=3 (naïve degree of divergence $\Delta = 10 \leftrightarrow \partial^2 F^4$) in which case the higher dimensional case also has a divergence but the lower dimensional case doesn't.

- In the SYM case, one has to distinguish between single-trace operators $\mathrm{tr}(\partial^2 F^4)$, for which there *are* divergences in both D=7 and D=6 cases, and double-trace operators $\mathrm{tr}(\partial F^2)\mathrm{tr}(\partial F^2)$ for which the D=6 divergence is *absent*.
- This apparently similar pair of double-trace D=7 and D=6 max. SYM candidate counterterms, with only the higher-dimension counterterm actually occuring with an infinite coefficient, looks very similar to the max. supergravity pair of candidates at D=6, L=3 (infinity occurs) and D=5, L=4 (infinity does not occur).

Algebraic Renormalization

Dixon; Howe, Lindstrom & White; Piguet & Sorella; Hennaux; Stora; Baulieu & Bossard

- Another approach to analyzing the divergences in supersymmetric gauge theories begins with the Callan-Symanzik equation for the renormalization of the Lagrangian as a operator insertion, governing, *e.g.*, mixing with the half-BPS operator $S^{(4)} = \mathrm{tr}(F^4)$. Letting the classical action be $S^{(2)}$, the C-Z equation in dimension D is $\mu \frac{\partial}{\partial \mu}[S^{(2)} \cdot \Gamma] = (4 - D)[S^{(2)} \cdot \Gamma] + \gamma_{(4)} g^{2n_{(4)}}[S^{(4)} \cdot \Gamma] + \cdots$, where $n_{(4)} = 4,\ 2,\ 1$ for $D = 5,\ 6,\ 8$.
- From this one learns that $(n_{(4)} - 1)\beta_{(4)} = \gamma_{(4)}$ so the beta function for the $S^{(4)} = \mathrm{tr}(F^4)$ operator is determined by the anomalous dimension $\gamma_{(4)}$.

- Combining the supersymmetry generator with a commuting spinor parameter to make a scalar operator $\mathcal{Q} = \bar{\varepsilon}Q$, the expression of SUSY invariance for a D-form density in D-dimensions is $\mathcal{Q}\mathcal{L}_D + d\mathcal{L}_{D-1} = 0$. Combining this with the SUSY algebra $\mathcal{Q}^2 = -i(\bar{\varepsilon}\gamma^\mu\varepsilon)\partial_\mu$ and using the Poincaré Lemma, one finds $i_{i(\bar{\varepsilon}\gamma\varepsilon)}\mathcal{L}_D + S_{(\mathcal{Q})|\Sigma}\mathcal{L}_{D-1} + d\mathcal{L}_{D-2} = 0$.
- Hence, one can consider cocycles of the extended nilpotent differential $d + S_{(\mathcal{Q})|\Sigma} + i_{i(\bar{\varepsilon}\gamma\varepsilon)}$ acting on formal form-sums $\mathcal{L}_D + \mathcal{L}_{D-1} + \mathcal{L}_{D-2} + \cdots$.
- The supersymmetry Ward identities then imply that the whole cocycle must be renormalized in a coherent way. In order for an operator like $S^{(4)}$ to mix with the classical action $S^{(2)}$, their cocycles need to have the same structure.

Ectoplasm

Gates, Grisaru, Knut-Whelau, & Siegel
Berkovits and Howe
Bossard, Howe & KSS

- The construction of supersymmetric invariants is isomorphic to the construction of cohomologically nontrivial closed forms in superspace:
  $I = \int_{M_0} \sigma^* \mathcal{L}_D$ is invariant (where σ^* is a pull-back to the "body" subspace M_0) if $\mathcal{L}_D$ is a closed form in superspace, and is nonvanishing if $\mathcal{L}_D$ is nontrivial.
- Revisit the BRST formalism, but now include all gauge symmetries (in particular including spatial diffeomorphisms) in the nilpotent BRST operator s. The invariance condition for $\mathcal{L}_D$ is $s\mathcal{L}_D + d_0\mathcal{L}_{D-1} = 0$ where d_0 is the usual bosonic exterior derivative. Since $s^2 = 0$ and s anticommutes with d_0, one obtains $s\mathcal{L}_{D-1} + d_0\mathcal{L}_{D-2} = 0$.

- So the cohomological problem reappears in BRST guise, but with the commuting spinor ε replaced by the commuting supersymmetry ghost. One needs to study the cohomology of the nilpotent operator $\delta = s + d_0$, whose cochains $\mathcal{L}_{D-q,q}$ are (D-q) forms with ghost number q, *i.e.* (D-q) forms with q spinor indices. The spinor indices are totally symmetric since the supersymmetry ghost is commuting.

- For gauge-invariant supersymmetric integrands, this establishes an isomorphism between the cohomology of closed forms in superspace (aka "ectoplasm") and the constuction of BRST invariant counterterms.

Superspace cohomology

Bonora, Pasti & Tonin

- Flat superspace has a standard basis of invariant 1-forms
$$E^a = dx^a - \frac{i}{2} d\theta^\alpha (\Gamma^a)_{\alpha\beta}\theta^\beta$$
$$E^\alpha = d\theta^\alpha$$
dual to which are the superspace covariant derivatives (∂_a, D_α)

- There is a natural bi-grading of superspace forms into even and odd parts: $\Omega^n = \oplus_{n=p+q}\Omega^{p,q}$

- Correspondingly, the flat superspace exterior derivative splits into three parts with bi-gradings (1,0), (0,1) & (-1,2):
$$d = d_0(1,0) + d_1(0,1) + t_0(-1,2)$$
bosonic der. fermionic der. torsion
$$d_0 \leftrightarrow \partial_\mu \qquad d_1 \leftrightarrow D_\alpha$$
where for a (p,q) form in flat superspace, one has
$$(t_o\omega)_{a_2\cdots a_p\beta_1\cdots\beta_{q+2}} \sim (\Gamma^{a_1})_{(\beta_1\beta_2}\omega_{a_1\cdots a_p\beta_3\cdots\beta_{q+2})}$$

- The nilpotence of the total exterior derivative d implies the relations

$$\begin{aligned} t_0^2 &= 0 \\ t_0 d_1 + d_1 t_0 &= 0 \\ d_1^2 + t_0 d_0 + d_0 t_0 &= 0 \end{aligned}$$

- Then, since $d\mathcal{L}_D = 0$, the lowest dimension nonvanishing cochain (or "generator") $\mathcal{L}_{D-q,q}$ must satisfy $t_0 \mathcal{L}_{D-q,q} = 0$, so $\mathcal{L}_{D-q,q}$ belongs to the t_0 cohomology group $H_t^{D-q,q}$.

- Starting with the t_0 cohomology groups $H_t^{p,q}$, one then defines a spinorial exterior derivative $d_s \; : \; H_t^{p,q} \rightarrow H_t^{p,q+1}$ by $d_s[\omega] = [d_1 \omega]$, where the [ ] brackets denote H_t classes.

Cederwall, Gran, Nilsson & Tsimpis
Howe & Tsimpis

- One finds that d_s is nilpotent, $d_s^2 = 0$, and so one can define spinorial cohomology groups $H_s^{p,q} = H_{d_s}(H_t^{p,q})$.

The groups $H_s^{0,q}$ give multi *pure spinors.*

- This formalism gives a way to reformulate the algebraic renormalization cohomology in terms of spinorial cohomology. The lowest dimension cochain, or *generator*, of a counterterm's superform will be d_s closed, *i.e.* it must be an element of $H_s^{D-q,q}$.

- Solving $d_s[\mathcal{L}_{D-q,q}] = 0$ then allows one to solve for all the higher components of $\mathcal{L}_D$ in terms of $\mathcal{L}_{D-q,q}$.

- To see how this formalism works, consider N=1 supersymmetry in D=10. Corresponding to the κ symmetries of strings and 5-branes, we have the D=10 Gamma matrix identities $t_0\Gamma_{1,2} = 0 \qquad t_0\Gamma_{5,2} = 0$.

- The second of these is relevant to the construction of *d*-closed forms in D=10. One may have a generator

$$L_{5,5} = \Gamma_{5,2} M_{0,3}$$

where $d_s[M_{0,3}] = 0$. The simplest example of such a form corresponds to a full superspace integral over *S*:

$$M_{\alpha\beta\gamma} = T_{\alpha\beta\gamma,\delta_1\cdots\delta_5}(D^{11})^{\delta_1\cdots\delta_5} S$$

where $T_{\alpha\beta\gamma,\delta_1\cdots\delta_5}$ is constructed from the D=10 Gamma matrices; it is totally symmetric in $\alpha\beta\gamma$ and totally antisymmetric in $\delta_1\cdots\delta_5$.

- Under dimensional reduction, closed D forms reduce to closed (D-1) forms, so one obtains directly the sequence of cocycles corresponding to non-BPS invariants in 4<D<10 dimensions, with generators $L_{D-5,5} \sim \Gamma_{D-5,2} M_{0,3}$.

- Now consider the cocycle of the D=10 SYM Lagrangian itself. This is an example of a Chern-Simons form, based on the closed 11-form $W_{11} = \Gamma_{5,2} \mathrm{tr} F^2$. In standard CS fashion, this can be written as $W_{11} = dZ_{10}$ where Z_{10} is a potential form in D=10, but it also has the property that it can be written as $W_{11} = dK_{10}$ where K_{10} is gauge invariant; its lowest component is $K_{8,2}$ .Thus, $K_{10} - Z_{10}$ is closed and so can be used to construct an integrated invariant.

- The 10-form Z_{10} can be taken to be $\Gamma_{5,2}Q_3$ where Q_3 is the Chern-Simons 3-form, $dQ_3 = \mathrm{tr}F^2$.

- One finds that the lowest dimension cochain in the D=10 Lagrangian cocycle has structure $\mathcal{L}_{5,5} = \Gamma_{5,2}Q_{0,3}$, *i.e.* it is of the *same* structure as that for the a full superspace integral counterterm.

- Consequently, full superspace integral cocycles have the same structure as that of the Lagrangian cocycle and thus are *not* subject to a nonrenormalization theorem.

- An example of a candidate counterterm which is allowed by this analysis is the full-superspace integral of the Konishi operator $\mathrm{tr}(W_r W_r)$. This is relevant to the *single*-trace divergences in L=2, D=7 max SYM. When evaluated on-shell, this full-superspace expression integrates to zero in the abelian case, but becomes a combination of $\mathrm{tr}(\partial^2 F^4)$ and $\mathrm{tr}(F^5)$ in the non-abelian case.

- Examples of operators that *are* ruled out by the ectoplasm/algebraic renormalization analysis are any half-BPS counterterms, such as the $\mathrm{tr}(F^4)$ or $(\mathrm{tr}(F^2))^2$ counterterms. In D dimensions, the generator component of such a 1/2 BPS cocycle is an (0,D) form of dimension 8-D/2. Since the structure of this cocycle is different from than that of the Lagrangian, the corresponding 1/2 BPS counterterm is *illegal*.

Double-trace SYM non-renormalization

- Similar analysis of the D=7 $\mathrm{tr}(\partial F^2)\mathrm{tr}(\partial F^2)$ L=2 double-trace candidate shows that its lowest cocycle components may be removed by the addition of exact terms, consistent with the D=7 SU(2) R-symmetry, thus leaving a (2,5) lowest dimension form like that of the classical Lagrangian. Thus, this structure is *not* protected.
Bossard, Howe & K.S.S
- In D=6, however, the situation is different. The R-symmetry is now SU(2)xSU(2) and one finds that there is *no* trivial term that can be added to shorten the D=6 double-trace cocycle so as to agree with the D=6 Lagrangian cocycle structure. Thus, the double-trace L=3 counterterm is *ruled out* in D=6.

Current outlook

- No mysteries persist in max. SYM: field-theoretic non-renormalization theorems explain all current calculational results.
- Providing the D=6, L=3 vs D=5, L=4 max. supergravity cases work similarly, the current SG calculational results would also be understood purely within field theory.
- In D=5 max. SYM, the L=6 double-trace counterterm should similarly be ruled out, and in D=4 max. SG, the L=5 and L=6 counterterms should also be illegal. So the first allowed D=4 max. SG counterterms would be full-superspace integrals at L=7. Rôle of duality? E_7 ->L=8?